

ON THE POISSON RELATION FOR COMPACT LIE GROUPS

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ABSTRACT. Intuition drawn from quantum mechanics and geometric optics raises the following long-standing question: can the length spectrum of a closed Riemannian manifold be recovered from its Laplace spectrum? The Poisson relation states that for any closed Riemannian manifold (M, g) the singular support of the trace of its wave group—a spectrally determined tempered distribution—is contained in the set consisting of $\pm\tau$, where τ is the length of a smoothly closed geodesic in (M, g) . Therefore, in cases where the Poisson relation is an equality, we obtain a method for retrieving the length spectrum of a manifold from its Laplace spectrum. The Poisson relation is known to be an equality for sufficiently “bumpy” Riemannian manifolds and there are no known counterexamples.

We demonstrate that the Poisson relation is an equality for a compact Lie group equipped with a generic bi-invariant metric. Consequently, the length spectrum of a generic bi-invariant metric (and the rank of its underlying Lie group) can be recovered from its Laplace spectrum. Furthermore, we exhibit a substantial collection \mathcal{G} of compact Lie groups—including those that are either tori, simple, simply-connected, or products thereof—with the property that for each group $U \in \mathcal{G}$ the Laplace spectrum of any bi-invariant metric g carried by U encodes the length spectrum of g and the rank of U . The preceding statements are special cases of results concerning compact *globally* symmetric spaces for which the semi-simple part of the universal cover is split-rank. The manifolds considered herein join a short list of families of non-“bumpy” Riemannian manifolds for which the Poisson relation is known to be an equality.

1. INTRODUCTION AND PROOF OF THE MAIN THEOREM

The (*Laplace*) *spectrum* of a closed Riemannian manifold (M, g) is the sequence $\lambda_0 = 0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k \nearrow +\infty$ consisting of the eigenvalues of its associated Laplace operator Δ_g , where each eigenvalue is repeated according to its multiplicity. A central problem in spectral geometry is to understand the extent to which the geometry of a Riemannian manifold can be recovered from its spectrum. While it is well known that the geometry of an arbitrary manifold cannot be completely recovered from its spectrum, the asymptotic expansion of the heat trace about its singularity at zero reveals that dimension, volume and certain integrals of local geometric invariants are spectrally determined. Inspired by intuitive arguments drawn from quantum mechanics and geometric optics, we have the following long-standing problem.

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Problem 1. The *length spectrum* of a Riemannian manifold (M, g) is the set, denoted by $\text{Spec}_L(M, g)$, consisting of the lengths of the manifold’s smoothly closed geodesics. Can the length spectrum of a Riemannian manifold be recovered from its Laplace spectrum?

There are currently no known examples of isospectral manifolds having distinct length spectra. In fact, isospectral manifolds arising from Sunada’s method necessarily have identical length spectra [Sun] (cf. [GorM, Theorem 1.3]).

A natural approach to obtaining a positive answer to this problem is to use a suitable trace formula. Indeed, using the so-called Poisson summation formula, and generalizations thereof, it has been demonstrated that Problem 1 can be answered positively for flat manifolds [MR] and Heisenberg manifolds [Pes]. And, through the Selberg trace formula, one can see that the length spectrum of a Riemann surface may be recovered from its Laplace spectrum [Hub1, Hub2, M]. A more general approach to this problem emerges from the work of Chazarain [Ch] and Duistermaat and Guillemin [DuGu] (cf. [CdV]) where it is proven that the singular support of the *trace of the wave group* of a manifold is contained in the set of periods of the geodesic flow (Section 1.2):

$$\text{SingSupp}(\text{Trace}(e^{-it\sqrt{\Delta_g}})) \subseteq \text{Spec}_L^\pm(M, g) \equiv \{\pm\tau : \tau \in \text{Spec}_L(M, g)\}.$$

This containment is referred to as the *Poisson relation* and, since the trace of the wave group of a manifold is a spectrally determined distribution, we arrive at the following intriguing question.

Problem 2. Is the Poisson relation an equality for all Riemannian manifolds?

It is clear that an affirmative answer to Problem 2 provides an affirmative answer to Problem 1. Hence, the remainder of this article will be concerned with understanding the extent to which Problem 2 has a positive answer.

For *clean* manifolds (Definition 1.2) it is in principle possible to use the trace formula of Duistermaat and Guillemin (Equation 1.5) to resolve whether the Poisson relation is an equality: the main obstacle is determining whether cancellations can occur in the trace formula (Section 1.2). By observing that sufficiently “bumpy” manifolds are clean and do not raise the specter of cancellation, Duistermaat and Guillemin were able to conclude that Problem 2 (and, consequently, Problem 1) has an affirmative answer for a generic Riemannian manifold [DuGu, p. 61]. It is also a straightforward consequence of their work that Problem 2 has a positive answer for compact rank-one symmetric spaces and, more generally, C_τ -manifolds (Section 1.3). In subsequent work, Duistermaat, Kolk and Varadarajan [DKV] (cf. [Gan]) answered Problem 2 affirmatively for compact *locally* symmetric spaces of the *non-compact type* (e.g., closed manifolds of constant negative sectional curvature) and it follows from a result of Guillemin that the Poisson relation is an equality for negatively curved manifolds [Gu, Theorem 4]. In establishing the last two statements the main chore is showing that the spaces are clean, while the issue of cancellations can be resolved in each case by well-known reasons (cf. [PR, Sec. 10]).

To the best of our knowledge, the preceding is an exhaustive account of the progress that has been made on Problems 1 and 2 prior to the results presented in this article. In particular, setting aside compact rank-one symmetric spaces and flat tori, there has been no previous progress on these questions regarding *globally* symmetric spaces, where a priori the occurrence of cancellations in the trace formula is a distinct possibility among the non-flat higher-rank symmetric spaces.

Compact Lie groups equipped with bi-invariant metrics form a natural and widely considered class of globally symmetric spaces (Section 1.3). After demonstrating that all globally symmetric spaces are clean (Theorem 1.8)—a property not enjoyed by every homogeneous space (Theorem 1.9)—we use the trace formula of Duistermaat and Guillemin to demonstrate that for any compact Lie group U the Poisson relation is an equality for a generic bi-invariant metric on U and, consequently, conclude that the length spectrum of a compact Lie group equipped with a generic bi-invariant metric can be recovered from its Laplace spectrum (Corollary 1.15 (1)). Furthermore, we observe that the spectrum of a generic bi-invariant metric encodes the rank of its underlying Lie group (Theorem 1.16). A more careful analysis allows us to exhibit an infinite collection \mathcal{G} of compact Lie groups with the property that for each $U \in \mathcal{G}$ the Laplace spectrum of any bi-invariant metric g supported by U encodes the length spectrum of g and the rank of U (Corollary 1.15 (2) and Theorem 1.16). The set \mathcal{G} properly contains the collection of groups that are either simple, simply-connected, tori or products thereof. The preceding statements are a strong indication that one should be able to recover the length spectrum of an arbitrary bi-invariant metric from its Laplace spectrum.

Corollary 1.15 is a special case of Theorem 1.13, which is a more general statement concerning compact symmetric spaces for which the non-Euclidean part of the universal cover is split-rank. As we will explain, the key to Theorem 1.13 is Theorem 1.23, which reveals that for certain symmetric spaces the Morse index modulo 4 of a closed geodesic depends only on the length τ of the geodesic and the dimension of the corresponding component of the fixed-point set of Φ_τ , the time- τ map of the geodesic flow.

Before reviewing the trace formula of Duistermaat and Guillemin in Section 1.2 and providing a precise statement of our results along with an outline of the supporting arguments in Sections 1.3 and 1.4, we pause briefly to provide the reader with a partial motivation for Problem 1 based on considerations in quantum mechanics.

1.1. Bohr's Correspondence Principle and the Length Spectrum. Consider a free particle in a Riemannian manifold (M, g) . Classical mechanics takes the viewpoint that the evolution of this particle is deterministic. Indeed, in classical mechanics the state space of the system is taken to be the co-tangent bundle T^*M equipped with the symplectic form ω_g induced by the metric g . Associated to the observable $p : T^*M \rightarrow \mathbb{R}$ given by $\zeta \mapsto \|\zeta\|_g^2$ is the Hamiltonian vector field X defined via the relationship

$$dp(\cdot) = \omega_g(X, \cdot),$$

which encodes Hamilton's equations of motion. The flow generated by X is the geodesic flow $\Phi : \mathbb{R} \times T^*M \rightarrow T^*M$ and its orbits describe the motion of the free particle under consideration.

On the other hand, quantum mechanics takes a probabilistic viewpoint. Indeed, the state space of the quantum system is taken to be $L^2(M, \nu_g)$, where ν_g is the measure induced by g , and an L^2 -normalized function $f \in L^2(M, \nu_g)$ is interpreted as a probability density; that is, for any measurable set $E \subset M$ the quantity $\int_E |f(x)|^2 d\nu_g$ represents the probability that the state of the particle is in the set E . The evolution of a probability density $f \in L^2(M, \nu_g)$ is governed by Schrödinger's equation:

$$\begin{cases} i\hbar \frac{\partial}{\partial t} \Psi(t, x) = -\hbar^2 \frac{1}{2} \Delta_g \Psi(t, x) \\ \Psi(0, x) = f(x) \end{cases}$$

whose solution is given by $\Psi(t, x) = e^{iht \frac{1}{2} \Delta_g} f(x)$. Therefore, the *Schrödinger flow* $S_g^h(t) \equiv e^{iht \frac{1}{2} \Delta_g}$, is the quantum-mechanical analogue of the geodesic flow and, via the functional calculus, we see that the Schrödinger flow is completely determined by the spectrum of (M, g) .

Now, the correspondence principle is the assertion that as $\hbar \rightarrow 0$ (i.e., as the “characteristic action” of the system becomes large relative to \hbar) the quantum dynamical system will converge (in some sense) to the classical system; so that, for \hbar small, the quantum system will reflect salient features of the corresponding classical dynamical system. As the periods of periodic orbits are a fundamental feature of the geodesic flow (i.e., the classical dynamics) and the spectrum of a Riemannian manifold determines the Schrödinger flow (i.e., quantum dynamics) it is natural to wonder whether the length spectrum can be recovered from its Laplace spectrum.

Mathematical motivation for Problem 1 can be found in the work of Colin de Verdière [CdV] and Duistermaat and Guillemin [DuGu] where it is demonstrated that for a generic (i.e., sufficiently “bumpy”) manifold the spectrum determines the Laplace spectrum. As it is germane to our results (Section 1.3), we now summarize the approach taken by Duistermaat and Guillemin, which utilizes $\sqrt{\Delta_g}$ rather than the semi-classical parameter \hbar used above.

1.2. The Trace Formula and “Bumpy” Metrics. The *wave group* of a Riemannian manifold (M, g) is the family of unitary operators $U_g(t) \equiv e^{-it\sqrt{\Delta_g}}$. The operators $U_g(t)$ are not trace class; however, for any Schwarz function $f(t)$, the operator $U_f \equiv \int_{-\infty}^{\infty} f(t) U_g(t) dt$ is of trace class. We then define the *trace of the wave group* $\text{Trace}(U_g(t))$ to be the tempered distribution:

$$f \in \mathcal{S}(\mathbb{R}) \mapsto \text{Trace}(U_f) \in \mathbb{R}.$$

The distribution kernel of the wave group of (M, g) is given by

$$U_g(t, x, y) = \sum_{j=0}^{\infty} e^{-it\sqrt{\lambda_j}} \varphi_j(x) \overline{\varphi_j(y)},$$

where $\{\varphi_j\}_{j=0}^\infty$ is an orthonormal basis of Δ_g -eigenfunctions with $\Delta_g \varphi_j = \lambda_j \varphi_j$. It follows that

$$\text{Trace}(U_g(t)) = \int_M U_g(t, x, x) d\nu_g = \sum_{j=0}^\infty e^{-it\sqrt{\lambda_j}},$$

which is the Fourier transform of the “spectral distribution” $\sigma(t) = \sum_{j=0}^\infty \delta(t - \sqrt{\lambda_j})$. It has been demonstrated by Chazarain [Ch, Theorem I] and Duistermaat and Guillemin [DuGu, Corollary 1.2] that the singular support of the trace of the wave group, which we will denote by $\text{SingSupp}(\text{Trace}(U_g(t)))$, is a subset of the periods of the periodic orbits of the geodesic flow:

$$(1.1) \quad \text{SingSupp}(\text{Trace}(U_g(t))) \subseteq \text{Spec}_L^\pm(M, g) \equiv \{\pm\tau : \tau \in \text{Spec}_L(M, g)\}.$$

This containment is known as the *Poisson relation* and, in light of Problem 1, understanding whether Equation 1.1 is an equality for all manifolds is a fascinating question (Problem 2). For manifolds satisfying the so-called “clean intersection hypothesis,” the trace formula of Duistermaat and Guillemin provides a means through which we can hope to analyze this problem.

Definition 1.2. Let (M, g) be a closed Riemannian manifold and, for each $t \in \mathbb{R}$, let $\Phi_t(\cdot) = \Phi(t, \cdot)$ be the time- t map of the associated geodesic flow. A period $\tau \in \text{Spec}_L^\pm(M, g)$ of the geodesic flow is said to have a *clean fixed-point set* or to be *clean* if

- (1) the fixed-point set of Φ_τ , denoted $\text{Fix}(\Phi_\tau)$, is a disjoint union of finitely many closed manifolds $\Theta_1, \dots, \Theta_r$;
- (2) for each $u \in \text{Fix}(\Phi_\tau)$ the fixed point set of $D_u \Phi_\tau$ is precisely equal to $T_u \text{Fix}(\Phi_\tau)$.
That is, if $J(t)$ is a periodic Jacobi field along the geodesic γ_u , with $\gamma'_u(0) = u$, then $(J(0), J'(0)) \in T_u \text{Fix}(\Phi_\tau)$.

Equivalently, $\tau \in \text{Spec}_L^\pm(M, g)$ is to have a *clean fixed-point set* or to be *clean*, if $|\tau|$ is a non-degenerate critical value of the energy functional $E : \Omega(M, g) \rightarrow \mathbb{R}$ on the loop space of (M, g) . We will agree to say that a Riemannian manifold (M, g) has *clean geodesic flow* or is *clean*, if every $\tau \in \text{Spec}_L^\pm(M, g)$ is clean.

Remark 1.3. Clearly, $\tau \in \text{Spec}_L^\pm(M, g)$ is clean if and only if $-\tau$ is clean.

Under the assumption that the period $\tau \in \text{Spec}_L^\pm(M, g)$ has a clean fixed-point set, Duistermaat and Guillemin determined that there is an interval $I_\tau \subset \mathbb{R}$ for which $I_\tau \cap \text{Spec}_L(M, g) = \{\tau\}$ and such that on I_τ the wave trace can be expressed as a sum of compactly supported distributions

$$(1.4) \quad \text{Trace}(U_g(t)) = \beta^{\text{even}}(t - \tau) + \beta^{\text{odd}}(t - \tau),$$

where $\beta^{\text{even}}(x)$ (respectively, $\beta^{\text{odd}}(x)$) is a distribution determined by the even-dimensional (respectively, odd-dimensional) components of $\text{Fix}(\Phi_\tau)$ and whose only possible singularity

occurs at $x = 0$ [DuGu, Theorem 4.5]. Of particular interest is the fact that the Fourier transforms of $\beta^{\text{even}}(x)$ and $\beta^{\text{odd}}(x)$ are given by smooth functions $\alpha^{\text{even}}(s)$ and $\alpha^{\text{odd}}(s)$, respectively, possessing the following asymptotic expansions at infinity:

$$(1.5) \quad \alpha^{\bullet}(s) \underset{s \rightarrow +\infty}{\sim} \sum_{k=0}^{\infty} \text{Wave}_k^{\bullet}(\tau) s^{(D_{\bullet}-2k-1)/2},$$

where \bullet denotes “even” or “odd” and D_{\bullet} is the maximum taken over the dimensions of the \bullet -dimensional components of $\text{Fix}(\Phi_{\tau})$ (see [DuGu, Theorem 4.5]).

The coefficients $\text{Wave}_k^{\text{even}}(\tau)$ and $\text{Wave}_k^{\text{odd}}(\tau)$ in the asymptotic expansions above are complex numbers known as the k -th wave invariants of τ , and, in light of the definition of the wave group, these are spectral invariants of the Riemannian manifold (M, g) . The moral of the trace formula is that the faster α^{even} (resp. α^{odd}) decays at infinity the less singular $\beta^{\text{even}}(t - \tau)$ (respectively, $\beta^{\text{odd}}(t - \tau)$) is at τ . Consequently, the trace formula informs us that a period τ with a clean fixed-point set is in $\text{SingSupp}(\text{Trace}(U_g(t)))$ if and only if at least one of its wave invariants is non-zero. That is, $\text{Trace}(U_g(t))$ is smooth at a clean period τ if and only if the Fourier transform of its restriction to I_{τ} is asymptotic to zero at infinity. It will be useful for us to recall that, since the subprincipal symbol of $\sqrt{\Delta}$ is zero [DuGu, p.58], the 0-th wave invariants of a clean period τ are given by the following formula [DuGu, Equation 4.8]:

$$(1.6) \quad \text{Wave}_0^{\bullet}(\tau) = \left(\frac{1}{2\pi i} \right)^{\frac{D_{\bullet}-1}{2}} \sum_{\substack{j=1 \\ \dim \Theta_j = D_{\bullet}}}^r i^{-\sigma_j} \int_{\Theta_j} d\mu_j^{\tau},$$

where \bullet denotes “even” or “odd,” σ_j is equal to the Morse index (in the free loop space) of a closed geodesic of length $|\tau|$ with initial velocity in the component Θ_j and μ_j^{τ} is a canonical positive measure on the submanifold Θ_j , which we will refer to as the *Duistermaat-Guillemin measure* [DuGu, Theorem 4.5 and p. 69-70] (cf. [BPU, Appendix]). In general, if τ is clean, the k -th wave invariant of τ is of the form $\text{Wave}_k^{\bullet}(\tau) = \sum_{j=1}^r C_{k,j}^{\bullet}$, where \bullet denotes “even” or “odd” and $C_{k,j}^{\bullet} \in \mathbb{C}$ is a constant determined by the component Θ_j of $\text{Fix}(\Phi_{\tau})$.¹ It is clear from the shape of the wave-invariants that for a clean period τ the issue of whether it is in the singular support is resolved according to whether cancellations occur in each $\text{Wave}_k^{\bullet}(\tau)$.

Remark 1.7. For each period $\tau \in \text{Spec}_L^{\pm}(M, g)$ with a clean fixed-point set, the Duistermaat-Guillemin densities μ^{τ} and $\mu^{-\tau}$ associated to $\text{Fix}(\Phi_{\tau}) = \text{Fix}(\Phi_{-\tau})$ agree [BPU, Lemma A.3]. Therefore, we obtain $\text{Wave}_0^{\bullet}(\tau) = \text{Wave}_0^{\bullet}(-\tau)$ for each $\tau \in \text{Spec}_L^{\pm}(M, g)$.

As we noted earlier, a Riemannian manifold is clean if each period of the geodesic flow is clean. Duistermaat and Guillemin have demonstrated that “cleanliness” is a generic property

¹We note that Chazarain obtains similar results to those of Duistermaat and Guillemin; however, he does not provide the explicit formula for the leading term of the asymptotic expansions given in Equation 1.6.

among Riemannian manifolds [DuGu, p. 61] and that generically the 0-th wave invariant does not vanish.² Specifically, they argue that sufficiently “bumpy” metrics (cf. [A, An]) are clean and that up to orientation these metrics have exactly one geodesic of a given length, so that the issue of cancellation does not arise for such spaces. Consequently, for sufficiently “bumpy” metrics, the wave-invariant $\text{Wave}_0^{\text{odd}}(\tau)$ never vanishes, thereby, establishing that for a generic Riemannian manifold Problem 2 and, hence, Problem 1 can be answered affirmatively.

1.3. Can You Hear the Length Spectrum of a Symmetric Space? The antithesis of a “bumpy” manifold is a (globally) symmetric space. A priori, it is not clear that symmetric spaces are clean and, even in the event that they are clean, the threat of cancellations in the trace formula looms large as $\text{Fix}(\Phi_\tau)$ will consist of many components for higher-rank symmetric spaces, which is in sharp contrast with the sufficiently “bumpy” spaces discussed previously.³ However, it is clear that the compact rank-one symmetric spaces (i.e. S^n , $\mathbb{R}P^n$, $\mathbb{C}P^n$, $\mathbb{H}P^n$ and Ca^2), which, henceforth, will be referred to as CROSSes, are clean. This follows immediately from the fact that for a CROSS the non-trivial orbits of the geodesic flow are all periodic with common primitive period τ_0 . In Section 2.1 we address the case of higher-rank symmetric spaces by exploiting the fact that (1) all geodesics in a symmetric space can be conjugated into a maximal flat and (2) in a symmetric space all periodic Jacobi fields are restrictions of Killing fields, which establishes that all compact symmetric spaces are clean.

Theorem 1.8. *A compact globally symmetric space is clean.*

This theorem tells us that in addition to the heat invariants, we have the wave invariants at our disposal in addressing inverse spectral problems that involve symmetric spaces. One might expect that all homogeneous manifolds are clean; unfortunately, this is already false for left-invariant metrics on $\text{SO}(3)$ and S^3 .

Theorem 1.9. *Within the class of left-invariant naturally reductive metrics on $\text{SO}(3)$ (respectively S^3 , in which case these are the Berger metrics), the clean metrics contain a residual set; however, the collection of unclean metrics is dense and contains certain normal homogeneous metrics.*

As we will see through an explicit computation of the geodesic flow, every left-invariant naturally reductive metric on $\text{SO}(3)$ (respectively S^3) satisfies condition (1) of Definition 1.2. The geodesic flow of the unclean metrics possess periods τ for which condition (2) of Definition 1.2 is not met. Gornet has also discovered locally homogeneous metrics that fail to be clean for the same reason [Gt]. Therefore, cleanliness cannot be taken for granted even among “nice” metrics.

²Let $\mathcal{R}(M)$ denote the space of Riemannian metrics on M . We will say that a metric property is *generic*, if the set of all metrics having this characteristic contains a residual set.

³It is known that locally symmetric spaces of the non-compact type (e.g., Riemann surfaces) are clean [DKV], but in this case cancellations cannot occur in the trace formula since the Morse index of a closed geodesic is always zero in this setting.

Returning to CROSSes, since for each period τ of the geodesic flow the fixed-point set of Φ_τ is the entire unit tangent bundle, we see from Equation 1.6 that $\text{Wave}_0^{\text{odd}}(\tau) \neq 0$. Therefore, the Poisson relation is an equality for CROSSes and it follows that the length spectrum of a CROSS can be recovered from its spectrum by computing the singular support of the trace of its wave group. More generally, if (M, g) is any manifold in which all non-trivial geodesics are closed and have a common primitive period τ —such a Riemannian manifold is commonly referred to as a C_τ -manifold (cf. [Be1])—then the Poisson relation is an equality.

In light of the fact that the conjecture is true for CROSSes, it is natural to wonder whether it is valid for every compact symmetric space.

A compact irreducible symmetric space $(M = G/K, g)$ comes in one of two flavors:

- **Type I:** the isometry group of M is a compact simple Lie group (e.g., CROSSes and Grassmannians)
- **Type II:** M is a compact simple Lie group U equipped with a bi-invariant metric, in which case $G = U \times U$ and $K = \Delta U$.

In both cases the symmetric metric on M is, up to scaling, the *standard metric* g_0 induced by the restriction of the negative of the Killing form $B_{\mathfrak{g}}$ of \mathfrak{g} to an $\text{Ad}(K)$ -invariant complement of \mathfrak{K} in \mathfrak{g} , where \mathfrak{g} and \mathfrak{K} denote the Lie algebras of G and K , respectively.

In general, a compact symmetric space $(M = G/K, g)$ is of the form

$$(1.10) \quad M = \Gamma \backslash (M_0 \times \widetilde{M}_1 \times \cdots \times \widetilde{M}_q),$$

where M_0 is a compact torus, \widetilde{M}_j is a simply-connected compact irreducible symmetric space, the metric g is induced by the metric $h \times c_1 g_0^1 \times \cdots \times c_q g_0^q$ on $M_0 \times \widetilde{M}_1 \times \cdots \times \widetilde{M}_q$ for some choice of constants $c_j > 0$ and flat metric h on M_0 , and Γ is a discrete subgroup of the *center* of $M_0 \times \widetilde{M}_1 \times \cdots \times \widetilde{M}_q$ (see Section 2.4):

$$(1.11) \quad Z(M_0 \times \widetilde{M}_1 \times \cdots \times \widetilde{M}_q) = M_0 \times Z(M_1) \times \cdots \times Z(M_q).$$

After replacing M_0 with $M_0/(\Gamma \cap M_0)$ we will assume, henceforth, that $\Gamma \cap M_0$ is trivial. It then follows that in each dimension there are only finitely many homogeneity types of compact symmetric spaces (cf. [GS, Lemma 2.8]). The compact simply-connected space $\widetilde{M}_{\text{cpt}} \equiv \widetilde{M}_1 \times \cdots \times \widetilde{M}_q$ is the *non-Euclidean* part of the universal cover of M . The space M is said to be of *compact type* precisely when M_0 is trivial; i.e., M has no Euclidean factors. We note that the non-Euclidean part of the metric is Einstein precisely when $c_1 = c_2 = \cdots = c_q$ [Be2, Theorem 7.74]. Ignoring the metric g , we refer to Equation 1.10 as the *homogeneity type* of the symmetric space. Given a homogeneity type $M = \Gamma \backslash (M_0 \times \widetilde{M}_1 \times \cdots \times \widetilde{M}_q)$ of a compact symmetric space, the space of symmetric metrics on M is the finite dimensional space

$$(1.12) \quad \mathcal{R}_{\text{symm}}(M) \equiv \mathcal{S}^+(d) \times \mathbb{R}_+ \times \cdots \times \mathbb{R}_+,$$

where d is the dimension of the torus M_0 and $\mathcal{S}^+(d)$ is the space of positive definite real symmetric $d \times d$ -matrices. We will let $\mathcal{P}(M)$ be the set of metrics in $\mathcal{R}_{\text{symm}}(M)$ for which the Poisson relation (Equation 1.1) is an equality and we let

$$\mathcal{W}_0(M) \subseteq \mathcal{P}(M)$$

denote the collection of metrics g such that for each period τ of the geodesic flow the leading term, $\text{Wave}_0^{\text{lead}}(\tau) \neq 0$, of the asymptotic expansion given in Equation 1.5 is non-zero.

Finally, we recall that a symmetric space $M = G/K$ is said to be *split-rank* if $\text{rank } G = \text{rank } M + \text{rank } K$ or, equivalently, the restricted roots of M (see Section 2.2) all have even multiplicity [Lo, Theorem VI.4.3]. By the classification of symmetric spaces the irreducible compact simply-connected split-rank spaces are the simple Lie groups, spaces of type A_{2n+1}^H ($n \geq 1$) and the exceptional space $E_{6(-36)}$, where we have adopted the notation of Loos (cf. [Lo, Theorem VI.4.4 and Tables 4 & 8]), and all compact split-rank spaces are finitely covered by products of these irreducible factors.

With these preliminaries out of the way, we now state our main result.

Theorem 1.13. *Let $M = \Gamma \backslash (M_0 \times \widetilde{M}_1 \times \cdots \times \widetilde{M}_q)$ be the homogeneity type of a compact symmetric space, where $\widetilde{M}_{\text{cpt}} \equiv \widetilde{M}_1 \times \cdots \times \widetilde{M}_q$ is trivial or split-rank. Also, for $j = 1, \dots, q$, let π_j be the natural projection of $M_0 \times Z(\widetilde{M}_1) \times \cdots \times Z(\widetilde{M}_q)$ onto its j -th factor. Then the following hold.*

- (1) $\mathcal{W}_0(M) \subseteq \mathcal{P}(M)$ contains a residual set. Consequently, for a generic symmetric metric on M , the Poisson relation is an equality and its length spectrum can be recovered from the Laplace spectrum of the metric.
- (2) Let \mathcal{H} be the collection of homogeneity types M , as above, such that:
 - (a) M is irreducible (e.g., simple Lie group), or
 - (b) M is not irreducible and the subgroup $\Gamma \leq M_0 \times Z(\widetilde{M}_1) \times \cdots \times Z(\widetilde{M}_q)$ is trivial or, for $j = 1, \dots, q$, satisfies the following:
 - (i) if $\widetilde{M}_j = \text{SU}(n+1)$, where $n \equiv 1 \pmod{2}$, then $\pi_j(\Gamma)$ is a proper subgroup of $Z(\text{SU}(n+1)) \simeq \mathbb{Z}_{n+1}$;
 - (ii) if $\widetilde{M}_j = \text{Spin}(2n+1)$, then $\pi_j(\Gamma)$ is the trivial subgroup in $Z(\text{Spin}(2n+1)) \simeq \mathbb{Z}_2$;
 - (iii) if $\widetilde{M}_j = \text{Sp}(n)$, where $n \equiv 1, 2 \pmod{4}$, then $\pi_j(\Gamma)$ is the trivial subgroup in $Z(\text{Sp}(n)) \simeq \mathbb{Z}_2$;
 - (iv) if $\widetilde{M}_j = \text{Spin}(2n)$, where $n \equiv 2 \pmod{4}$, then $\pi_j(\Gamma)$ is trivial or $\mathbb{Z}_2 \oplus 1$ in $Z(\text{Spin}(2n)) \simeq \mathbb{Z}_2 \oplus \mathbb{Z}_2$;
 - (v) if $\widetilde{M}_j = \text{Spin}(2n)$, where $n \equiv 3 \pmod{4}$, then $\pi_j(\Gamma)$ is trivial or \mathbb{Z}_2 in $Z(\text{Spin}(2n)) \simeq \mathbb{Z}_4$;
 - (vi) if $\widetilde{M}_j = E_7$, then $\pi_j(\Gamma)$ is trivial in $Z(E_7) \simeq \mathbb{Z}_2$;
 - (vii) if \widetilde{M}_j is any other irreducible simply-connected split-rank space, then there are no restrictions on $\pi_j(\Gamma)$.

If $M \in \mathcal{H}$, then $\mathcal{W}_0(M)$ equals $\mathcal{R}_{\text{symm}}(M)$ and we conclude that the Poisson relation is an equality for any symmetric metric on M . Consequently, if M is in \mathcal{H} , then for any metric $g \in \mathcal{R}_{\text{symm}}(M)$ Poisson relation is an equality and the length spectrum of g can be recovered from its Laplace spectrum.

- (3) If M is split-rank; i.e., M_0 is trivial, then $\mathcal{W}_0(M)$ contains the Einstein metric induced by $c(g_0^1 \times \cdots \times g_0^q)$, for each $c > 0$.

A compact Lie group U equipped with a bi-invariant metric is a symmetric space having homogeneity type:

$$(1.14) \quad U = \Gamma \backslash (T \times \tilde{U}_1 \times \cdots \times \tilde{U}_q),$$

where T is a torus, the factors $\tilde{U}_1, \dots, \tilde{U}_q$ are simply-connected simple Lie groups and Γ is a discrete subgroup of $Z(U) = T \times Z(\tilde{U}_1) \times \cdots \times Z(\tilde{U}_q)$, the center of U . Thus, we immediately obtain the following specialization of the previous result.

Corollary 1.15. *Let $U = \Gamma \backslash (T \times \tilde{U}_1 \times \cdots \times \tilde{U}_q)$ be a compact Lie group and let $\mathcal{R}_{\text{bi}}(U) = \mathcal{R}_{\text{symm}}(U)$ denote the associated space of bi-invariant metrics on U . Additionally, let \mathcal{G} be the collection of compact Lie groups in \mathcal{H} ; that is, \mathcal{G} consists of all compact Lie groups for which Γ is trivial or satisfies certain constraints depending on $\tilde{U}_1 \times \cdots \times \tilde{U}_q$, as described in Theorem 1.13(2). Then the following hold.*

- (1) $\mathcal{W}_0(U) \subseteq \mathcal{P}(U)$ contains a residual set. Consequently, for a generic bi-invariant metric on U the Poisson relation is an equality and its length spectrum can be recovered from its Laplace spectrum.
- (2) If U is in \mathcal{G} , then $\mathcal{W}_0(U)$ equals $\mathcal{R}_{\text{bi}}(U)$ and we conclude that the Poisson relation is an equality for every bi-invariant metric supported by U . Consequently, if U is a member of \mathcal{G} , then the length spectrum of any bi-invariant metric $g \in \mathcal{R}_{\text{bi}}(U)$ can be recovered from its Laplace spectrum.
- (3) If U is semi-simple, then $\mathcal{W}_0(U) \subseteq \mathcal{P}(U)$ contains the (unique up to scaling) bi-invariant Einstein metric induced by the negative of the Killing form on the Lie algebra of U .

Therefore, we obtain a positive answer to Problem 2 and, consequently, Problem 1 for a substantial portion of bi-invariant metrics. In particular, we see that Problem 2 has an affirmative answer for every bi-invariant metric on a compact Lie group that is simple, simply-connected, a torus or a product thereof.

We conclude this section with an observation—proven in Section 2.2—suggesting that the rank of a compact Lie group is encoded in the spectrum of any of its associated bi-invariant metrics.

Theorem 1.16. *The rank of a compact Lie group U is encoded in the spectrum of any bi-invariant metric $g \in \mathcal{W}_0(U)$. Consequently, within $\mathcal{W}_0^{\text{Lie}} \equiv \cup_U \mathcal{W}_0(U)$, where the union is over all compact Lie groups, rank is a spectral invariant.*

1.4. Proof of Theorem 1.13. We present the proof of Theorem 1.13 modulo some technical details that will be addressed in Section 3 after we have reviewed further aspects of symmetric spaces in Section 2.

Proof of Theorem 1.13. Theorem 1.8 states that compact symmetric spaces are clean; therefore, our strategy is to show that for each $\tau \in \text{Spec}_L^\perp(M, g)$, the leading term, $\text{Wave}_0^{\text{lead}}(\tau)$, in Equation 1.5 is non-zero. By Remark 1.7, it is enough to show this for $\tau \in \text{Spec}_L(M, g)$; i.e., the lengths of the closed geodesics.

Let (M, g) be a compact symmetric space without any further restrictions, for now, and let $\tau \in \text{Spec}_L(M, g)$. Since τ is clean, by Theorem 1.8, we know that $\text{Fix}(\Phi_\tau)$ is a disjoint union of finitely many closed submanifolds $\Theta_1, \dots, \Theta_r$ in the unit tangent bundle of M . If we let D denote the dimension of $\text{Fix}(\Phi_\tau)$ and let \bullet denote “even” or “odd” according to the parity of D , then by Equation 1.6 we have

$$\text{Wave}_0^{\text{lead}}(\tau) = \text{Wave}_0^\bullet(\tau) = \left(\frac{1}{2\pi i} \right)^{\frac{D-1}{2}} \sum_{\substack{j=1 \\ \dim \Theta_j = D}}^r i^{-\sigma_j} \int_{\Theta_j} d\mu_j^\tau.$$

Since the measures μ_j^τ are positive, the vanishing of $\text{Wave}_0^\bullet(\tau)$ depends in part on the value of the Morse indices, which can be difficult to compute for a general Riemannian manifold. However, as we note in Section 2.3, since (M, g) is a compact symmetric space the Morse index of a closed geodesic in (M, g) can be computed in terms of the restricted roots of the symmetric space [Z3]. From this we deduce (see Equation 2.11) that for a *non-trivial* closed geodesic $\gamma : [0, 1] \rightarrow (M, g)$, the Morse index has the following form:

$$(1.17) \quad \sigma_{\Delta M}(\gamma) = F_{(M, g)}(\gamma'(0)) - \dim \text{Fix}_\gamma(\Phi_\tau) + \dim M,$$

where $\text{Fix}_\gamma(\Phi_\tau)$ is the component of $\text{Fix}(\Phi_\tau) \subseteq SM$ containing $\gamma'(t)/\|\gamma'(t)\|$.

It will be useful to observe that the Morse index of a closed geodesic in certain symmetric spaces is influenced by the lengths of the components of its velocity vector (in the non-Euclidean factor).

Proposition 1.18. *Let $M = \Gamma \backslash (M_0 \times \widetilde{M}_1 \times \dots \times \widetilde{M}_q)$ be the homogeneity type of a compact symmetric space, where $\widetilde{M}_{\text{cpt}} = \widetilde{M}_1 \times \dots \times \widetilde{M}_q$ is split-rank, and for $j = 1, \dots, q$, let $M_j \equiv \widetilde{M}_j / \Gamma_j$, where Γ_j is the projection of Γ onto its j -th factor. Now, consider the symmetric metric g on M induced by the metric $h \times \tilde{g}_1 \times \dots \times \tilde{g}_q$ and, for $j = 1, \dots, q$, let g_j be the metric on M_j induced by \tilde{g}_j . Then there is a function $f_{(M, g)}(x_0, x_1, \dots, x_q) = \sum_{j=1}^q f_{(M_j, g_j)}(x_j)$, where $f_{(M_j, g_j)}(0) = 0$ for each $j = 1, \dots, q$, such that if $\gamma : [0, 1] \rightarrow (M, g)$ is a non-trivial closed geodesic with $\gamma'(0) \equiv (v_0, v_1, \dots, v_q)$, then*

$$(1.19) \quad F_{(M, g)}(\gamma'(0)) \equiv f_{(M, g)}(\|v_0\|, \|v_1\|, \dots, \|v_q\|) \pmod{4}.$$

Consequently, the function

$$h_{(M,g)}(x_0, x_1, \dots, x_q, y) = \begin{cases} f_{(M,g)}(x_0, x_1, \dots, x_q) - y + \dim M & , \text{ when } x_j \neq 0 \text{ for some } j \\ 0 & , \text{ otherwise} \end{cases}$$

satisfies

$$(1.20) \quad \sigma_{\Delta M}(\gamma) \equiv h_{(M,g)}(\|v_0\|, \|v_1\|, \dots, \|v_q\|, \dim \text{Fix}_\gamma(\Phi_\tau)) \pmod{4}.$$

Proof. See Section 3.1. □

With an eye towards establishing that the Poisson relation holds for a generic split-rank symmetric space we make the following definition.

Definition 1.21. Let (N, g) be a closed Riemannian manifold with universal cover $(\tilde{N}, \tilde{g}) = (\tilde{N}_1 \times \dots \times \tilde{N}_k, \tilde{g}_1 \times \dots \times \tilde{g}_k)$. We will say the length spectrum of (N, g) is *component length unique* (CLU) if for any $\tau \in \text{Spec}_L(N, g)$, there are nonnegative constants $c_1(\tau), \dots, c_k(\tau)$ such that for any closed geodesic $\gamma : [0, 1] \rightarrow (N, g)$ of length τ the velocity vector $\gamma'(0) = (v_1, \dots, v_k)$ satisfies $\|v_j\|^2 = c_j(\tau)$ for $j = 1, \dots, k$. That is, the length of the closed geodesic γ determines the lengths of the components of $\gamma'(t)$.

Lemma 1.22. Let $M = \Gamma \backslash (M_0 \times \tilde{M}_1 \times \dots \times \tilde{M}_q)$ have the homogeneity type of a compact symmetric space. The set of symmetric metrics on M with CLU length spectrum is a residual set in $\mathcal{R}_{\text{symm}}(M)$.

Proof. See Section 3.2. □

The following proposition tells us that for many compact symmetric spaces having a non-Euclidean part that is split-rank, the Morse index of a closed geodesic γ is determined modulo 4 by its length τ and the dimension of its corresponding component in $\text{Fix}(\Phi_\tau)$ (cf. [W, Theorem 3 and Corollary 3.6]).

Theorem 1.23. Let (M, g) be a compact symmetric space, where \tilde{M}_{cpt} is split-rank, that satisfies any one of the following:

- (1) the length spectrum of (M, g) is component length unique,
- (2) $M \in \mathcal{H}$, where \mathcal{H} is defined as in Theorem 1.13,
- (3) the universal cover of M has no Euclidean part and g is the unique up to scaling G -invariant Einstein metric on M .

Then, there is a function $H_{(M,g)} : \text{Spec}_L(M, g) \times \{0, 1, \dots, 2 \cdot \dim M - 1\} \rightarrow \{0, 1, 2, 3\}$ such that the Morse index of any closed geodesic γ in (M, g) of length τ satisfies

$$(1.24) \quad \sigma_{\Delta M}(\gamma) \equiv H_{(M,g)}(\tau, \dim \text{Fix}_\gamma(\Phi_\tau)) \pmod{4}.$$

In the event that (M, g) satisfies either (3) or (4) above, this relationship only depends on $\dim(\text{Fix}_\gamma(\Phi_\tau))$:

$$(1.25) \quad \sigma_{\Delta M}(\gamma) \equiv H_{(M,g)}(\dim \text{Fix}_\gamma(\Phi_\tau)) \pmod{4},$$

where $H_{(M,g)}(x) = x + C(M)$.

Proof. See Section 3.3. □

Now, let (M, g) be a compact symmetric space where $\widetilde{M}_{\text{cpt}}$, the non-Euclidean part of the universal cover of M , is split-rank. Suppose further that (M, g) satisfies any one of conditions (1)-(3) in Theorem 1.23, then it is clear from the conclusion of the theorem that for any $\tau \in \text{Spec}_L(M, g)$ we have $\text{Wave}_0^{\text{lead}}(\tau) = \text{Wave}_0^\bullet(\tau) \neq 0$, where \bullet equals “even” or “odd” according to the parity of $\dim(\text{Fix}(\Phi_\tau))$. That is, there are no cancellations in the leading term of the wave trace associated to τ . And, we conclude that τ is in the singular support of the wave trace and $\dim(\text{Fix}(\Phi_\tau))$ is a spectral invariant. In particular, for any symmetric space as in the statement of Theorem 1.23, the Poisson relation is an equality. As we also established in Lemma 1.22 that the symmetric metrics on M with CLU length spectrum are generic in $\mathcal{R}_{\text{symm}}(M)$, the theorem follows. □

Since the dimension of a Riemannian manifold is a spectral invariant and we know that for any non-zero τ in the length spectrum of a compact symmetric space M of rank at least 2 we have $\dim(\text{Fix}(\Phi_\tau)) < 2 \cdot \dim M - 1$, the following observation is an immediate consequence of Theorem 1.13.

Corollary 1.26. *Let $M = \Gamma \backslash (M_0 \times \widetilde{M}_1 \cdots \times \widetilde{M}_q)$ be the homogeneity type of a compact symmetric space where $\widetilde{M}_{\text{cpt}}$ is trivial or split-rank. Then a generic metric $g \in \mathcal{R}_{\text{symm}}(M)$ cannot be isospectral to a compact rank-one symmetric space or, more generally, a C_τ -manifold. In the case that M satisfies condition 2(a) or 2(b) of Theorem 1.13, then the preceding conclusion holds for every metric $g \in \mathcal{R}_{\text{symm}}(M)$.*

This suggests that, in general, the wave invariants might distinguish compact higher-rank symmetric spaces from compact rank-one symmetric spaces (cf. Theorem 1.16).

We round out this section by making the following observation, which follows directly from Lemma 1.22.

Theorem 1.27. *Let $M = \Gamma \backslash (M_0 \times \widetilde{M}_1 \times \cdots \times \widetilde{M}_q)$ be the homogeneity type of a compact symmetric space having the property that for every $g \in \mathcal{R}_{\text{symm}}(M)$, the function $F_{(M,g)}$ in Equation 1.17 associated with (M, g) satisfies Equation 1.19 for any closed geodesic $\gamma : [0, 1] \rightarrow (M, g)$. Then $\mathcal{P}(M)$ contains a residual set. Consequently, the length spectrum of a generic symmetric metric on M can be recovered from the Laplace spectrum of the metric.*

Theorem 1.13(1) is an instance of this result and we suspect that Theorem 1.27 is applicable to a broader class of symmetric spaces. However, in Example 3.16 we see that it cannot be applied to all symmetric spaces.

1.5. Concluding Remarks. To the best of our knowledge, the manifolds considered in Theorem 1.13 along with C_τ -manifolds (e.g., CROSSes), flat manifolds [MR], Heisenberg manifolds [Pes], compact locally symmetric spaces of the non-compact type (i.e., non-positively curved

locally symmetric spaces without a Euclidean factor) [Hub1, Hub2, M, Gan, DKV] (cf. [PR, Sec. 10]), *negatively* curved manifolds [Gu, Theorem 4] and, recently, certain lens spaces [Ci] are the only non-generic spaces for which it is known that one may recover the length spectrum directly from the Laplace spectrum. And, to date, there are no examples to the contrary. Indeed, as was noted earlier, all isospectral pairs arising from Sunada’s method must have identical length spectra [Sun, Section 4] (cf. [GorM, Theorem 1.3]).

In light of the above it is tempting to prove the conjecture for an arbitrary symmetric space by extending Theorem 1.23. However, Example 3.15 shows that this is not possible for compact Lie groups, in general, and Example 3.16 demonstrates that restricting our attention to irreducible symmetric spaces will not be successful. In both cases we find two closed geodesics γ_1 and γ_2 of the same length τ and for which $\dim \text{Fix}_{\gamma_1}(\Phi_\tau)$ and $\dim \text{Fix}_{\gamma_2}(\Phi_\tau)$ are equal; however, the difference between $\sigma_{\Delta M}(\gamma_1)$ and $\sigma_{\Delta M}(\gamma_2)$ is congruent to two modulo 4, which raises the possibility of cancellation in the trace formula. Therefore, in contrast with the case of locally symmetric spaces of the non-compact type, determining whether the Poisson relation is an equality for an arbitrary compact Lie group or, more generally, symmetric space will ostensibly require a better understanding of the Duistermaat-Guillemin measures associated to the components of $\text{Fix}(\Phi_\tau)$ (see p. 6). We will take this up in a separate article.

More generally, one might be curious about the extent to which this type of analysis can be carried out for compact Lie groups equipped with arbitrary left-invariant metrics and other homogeneous spaces. In Section 4 we observe that when considering left-invariant metrics on $\text{SO}(3)$ (respectively, S^3) we encounter two hurdles. First, not all left-invariant metrics are clean. As we noted in Theorem 1.9, we exhibit left-invariant naturally reductive metrics on $\text{SO}(3)$ (respectively, S^3) that are not clean. Therefore, being a “nice” left-invariant metric does not guarantee access to the trace formula of Duistermaat and Guillemin. Second, even among the clean left-invariant naturally reductive metrics on $\text{SO}(3)$ (respectively, S^3), it appears quite difficult to determine whether the leading term of the trace formula is non-zero for each length τ (cf. Proposition 4.33). These obstacles highlight the need for a different approach to the trace formula in the homogeneous setting.

Structure of the Paper. In Section 2, we prove Theorem 1.8, which establishes that compact symmetric spaces are clean, and we review Ziller’s method (see Theorem 2.9) for computing the Morse index of a closed geodesic in a symmetric space via the restricted roots. We also prove Theorem 1.16, which establishes that among generic bi-invariant metrics, rank is a spectral invariant. In Section 3, we complete the argument for Theorem 1.13 by proving Proposition 1.18, Lemma 1.22 and Theorem 1.23 which shows that for certain compact symmetric spaces, the Morse index modulo 4 of a closed geodesic γ of length τ only depends on τ and the dimension of the component of $\text{Fix}(\Phi_\tau)$ containing $\gamma'(0)$. The arguments in Section 3 rely on an explicit description of the co-root lattice, central lattice and lowest strongly dominant form associated to each of the indecomposable abstract root systems. This data is computed and catalogued in Appendix A. In Section 4, we present the proof of Theorem 1.9, which demonstrates that not

all homogeneous metrics are clean: the culprit will be the *Type II geodesics* (Definition 4.17). This argument is lengthy as we must explicitly compute the closed geodesics of a left-invariant naturally reductive metric on $\mathrm{SO}(3)$ and compute the Poincare map along these geodesics. We conclude the section by examining the Poisson relation for the clean left-invariant naturally reductive metrics on $\mathrm{SO}(3)$ and observe that resolving the cancellation issue in the leading term of the trace formula appears problematic for lengths arising from *Type III geodesics* (Propositions 4.33 and 4.34). Certain technical concerns related to naturally reductive metrics are relegated to Appendix B.

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2. COMPUTING THE MORSE INDEX IN A SYMMETRIC SPACE

The purpose of this section is to fix some notation, terminology and useful facts concerning symmetric spaces. In particular, we will demonstrate in Proposition 2.2 that for any τ in the length spectrum of a symmetric space, the fixed-point set $\mathrm{Fix}(\Phi_\tau)$ is a disjoint union of finitely many homogeneous manifolds. We will then use this to establish Theorem 1.8. In Theorem 2.9 we will recall a result of Ziller which states the Morse index of a closed geodesic γ in a *compact* symmetric space may be computed in terms of the restricted roots of the symmetric space and, upon closer inspection, we will notice in Equation 2.11 that this expression for the Morse index also involves $\dim(\mathrm{Fix}(\Phi_\tau))$, where τ is the length of the closed geodesic. Finally, we will recall the definition of the center of a symmetric space and remind the reader of the co-root, integral and central lattices of a symmetric space.

2.1. Geodesics and Cleanliness in a Symmetric Space. Let M be a symmetric space with G equal to the connected component of the identity in $\mathrm{Isom}(M)$ and $K = G_{p_0}$ the connected component of the isotropy group of a fixed point $p_0 \in M$, so that $M = G/K$. The Lie algebra \mathfrak{g} of G can be written as $\mathfrak{g} = \mathfrak{K} \oplus \mathfrak{p}$, where \mathfrak{K} is the Lie algebra of K and \mathfrak{p} is an $\mathrm{Ad}(K)$ -invariant complement. We then have the relations

$$[\mathfrak{K}, \mathfrak{p}] \subseteq \mathfrak{p}, [\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{K}, \text{ and } [\mathfrak{K}, \mathfrak{K}] \subseteq \mathfrak{K}$$

so that the linear transformation $s : \mathfrak{g} \rightarrow \mathfrak{g}$ which is the identity on \mathfrak{K} and negative the identity on \mathfrak{p} is a Lie algebra homomorphism known as the Cartan involution. The elements of \mathfrak{g} generate Killing vector fields on M by defining

$$X_q^* \equiv \frac{d}{dt} \Big|_{t=0} \exp_G(tX) \cdot q,$$

for any $q \in M$ and $X \in \mathfrak{g}$. It follows that \mathfrak{p} can be naturally identified with $T_p M$ via

$$X \in \mathfrak{p} \mapsto X_{p_0}^* \in T_{p_0} M.$$

Under this identification we see that the Levi-Civita connection and curvature tensor are given by:

$$(\nabla_{X^*} Y)(p_0) = [X^*, Y] \text{ and } R_{p_0}(X^*, Y^*)Z^* = -[[X, Y], Z]_{p_0}^*.$$

Of particular interest to us is the fact that the geodesics in M are the integral curves of the Killing vector fields and it follows that all self-intersections of a geodesic in M are smooth (cf. [?, Theorem 3.11]).

Now, let \mathfrak{a} be a maximal abelian subspace contained in \mathfrak{p} . All such subspaces are conjugate via $\text{Ad}(K)$ and their common dimension is referred to as the *rank* of M . Furthermore, we know that $\mathfrak{p} = \cup_{k \in K} \text{Ad}(k)\mathfrak{a}$ and $\mathfrak{a} \leq \mathfrak{p} \equiv T_{p_0} M$ exponentiates to a totally geodesic maximal flat $F_{\mathfrak{a}} \equiv \exp_M(\mathfrak{a}) \simeq T^m \times \mathbb{R}^d$ in M , where T^m is a flat m -torus and \mathbb{R}^d is d -dimensional euclidean space. Therefore, since every vector $v \in \mathfrak{p} \equiv T_{p_0} M$ can be conjugated into \mathfrak{a} , we see that every geodesic in M can be conjugated via the isometry group into the maximal flat $F_{\mathfrak{a}}$ in M . We then define the *integral lattice* of M to be

$$(2.1) \quad \Lambda_I(M) = \{v \in \mathfrak{a} : \exp_G(v) \cdot p_0 = p_0\} = \{v \in \mathfrak{a} : \exp_G(v) = e\}$$

Now, if τ is an element in the length spectrum of M , then there are finitely many closed geodesics in $F_{\mathfrak{a}}$ of length τ . It then follows that $\text{Fix}(\Phi_{\tau})$ is the disjoint union of finitely many homogeneous submanifolds $\Theta_1 = G \cdot v_1, \dots, \Theta_r = G \cdot v_r$, where $v_1, \dots, v_r \in \mathfrak{a}$ is a maximal collection of non-conjugate unit vectors such that $\gamma_j(t) \equiv \exp_G(tv_j) \cdot p_0$, $0 \leq t \leq \tau$, is a closed geodesic of length τ for each $j = 1, \dots, r$. In summary, we have the following.

Proposition 2.2. *Let M be a symmetric space and τ an element of its length spectrum. Then $\text{Fix}(\Phi_{\tau})$ is a disjoint union of finitely many homogeneous manifolds $\Theta_1, \dots, \Theta_r$, where for each $j = 1, \dots, r$, we have $\Theta_j = G \cdot \gamma_j'(0)$ for some unit speed closed geodesic γ_j of length τ with $\gamma_j'(0) \in \Theta_j$.*

With this in hand we may now prove Theorem 1.8.

Proof of Theorem 1.8. Let $M = G/K$ be a compact symmetric space and fix τ in the length spectrum of M . By Proposition 2.2, we see that $\text{Fix}(\Phi_{\tau})$ is a disjoint union of finitely many closed manifolds, where each component is of the form $G \cdot u$ for some $u \in \text{Fix}(\Phi_{\tau})$. Since M is a symmetric space, we know that the *periodic* Jacobi fields along any geodesic $\gamma(t)$ in M are of the form

$$J(t) = \frac{d}{ds}|_{s=0} \exp_G(sX) \cdot \gamma(t) \equiv X_{\gamma(t)}^*,$$

where $X \in \mathfrak{g}$. That is, all Jacobi fields in a (not necessarily compact) symmetric space are restrictions of Killing vector fields (cf. [Z3, proof of Theorem 2]). It then follows that for any $u \in \text{Fix}(\Phi_{\tau})$ we have $\text{Fix}(D_u \Phi_{\tau}) = T_u G \cdot u = T_u \text{Fix}(\Phi_{\tau})$. Therefore, τ is a clean length. \square

2.2. Restricted Root Systems. Now, let $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \oplus i\mathfrak{g}$ denote the complexification of \mathfrak{g} . Since $\mathfrak{a} \subset \mathfrak{p} \subset \mathfrak{g}$ is abelian we see that $\{\text{ad}(x) : x \in \mathfrak{a}\}$ is a commuting family of skew-adjoint linear transformations on $\mathfrak{g}_{\mathbb{C}}$. Hence, they are simultaneously diagonalizable. For any $\beta \in \mathfrak{a}^* \equiv \text{Hom}(\mathfrak{a}, \mathbb{R})$ we let $\mathfrak{g}_{\mathbb{C}}^{\beta}$ be the subspace of $\mathfrak{g}_{\mathbb{C}}$ defined by

$$\mathfrak{g}_{\mathbb{C}}^{\beta} = \{y \in \mathfrak{g}_{\mathbb{C}} : [x, y] = i\pi\beta(x)y \text{ for all } x \in \mathfrak{a}\}.$$

The set $R = R(M) \equiv \{\beta \neq 0 \in \mathfrak{a}^* : \mathfrak{g}_{\mathbb{C}}^{\beta} \text{ is non-trivial}\}$ is referred to as the set of *restricted roots of M* with respect to \mathfrak{a} . The restricted roots of M only depend on the universal cover \widetilde{M} of M and when \widetilde{M} has no Euclidean factor $R = R(M)$ is a root system of \mathfrak{a} (cf. Appendix A). In any event, we have the following decomposition.

$$\mathfrak{g}_{\mathbb{C}} = \oplus_{\beta \in R} \mathfrak{g}_{\mathbb{C}}^{\beta} = \mathfrak{g}_{\mathbb{C}}^0 \oplus \oplus_{\beta \in R} \mathfrak{g}_{\mathbb{C}}^{\beta}.$$

When M is not flat, the connected components of $\mathfrak{a} - \cup_{\beta \in R} \ker(\beta)$ are called Weyl chambers. A choice of a Weyl chamber \mathcal{C} leads to a decomposition $R = R^+ \cup R^-$ of the roots into positive and negative roots, where $R^+ = R^+(\mathcal{C}) \equiv \{\beta \in R : \beta > 0 \text{ on } \mathcal{C}\}$. For each positive root β we define the real vector space

$$\mathfrak{g}^{\beta} = \mathfrak{g} \cap (\mathfrak{g}_{\mathbb{C}}^{\beta} + \mathfrak{g}_{\mathbb{C}}^{-\beta})$$

and notice that it is s -invariant, since $s(\mathfrak{g}_{\mathbb{C}}^{\beta}) = \mathfrak{g}_{\mathbb{C}}^{-\beta}$ for each $\beta \in R$. The s -invariance of \mathfrak{g}^{β} implies that we have the decomposition

$$\mathfrak{g}^{\beta} = \mathfrak{K}^{\beta} + \mathfrak{p}^{\beta},$$

where $\mathfrak{K}^{\beta} = \mathfrak{K} \cap \mathfrak{g}^{\beta}$ and $\mathfrak{p}^{\beta} = \mathfrak{p} \cap \mathfrak{g}^{\beta}$. Also, since \mathfrak{a} is a maximal abelian subspace in \mathfrak{p} , we see that $\mathfrak{g}^0 = \mathfrak{g} \cap \mathfrak{g}_{\mathbb{C}}^0 = Z_{\mathfrak{g}}(\mathfrak{a}) = Z_{\mathfrak{K}}(\mathfrak{a}) + \mathfrak{a}$. One can check that

$$(2.3) \quad \mathfrak{p}^{\beta} = \{y \in \mathfrak{p} : \text{ad}(x)^2(y) = -\pi^2\beta(x)^2y \text{ for all } x \in \mathfrak{a}\}$$

$$(2.4) \quad \mathfrak{p} = \mathfrak{a} \oplus (\oplus_{\beta \in R^+} \mathfrak{p}^{\beta})$$

$$(2.5) \quad n_{\beta} \equiv \dim \mathfrak{p}^{\beta} = \dim \mathfrak{K}^{\beta},$$

where the integer n_{β} is said to be the *multiplicity* of β (in M). Then for any $v \in \mathfrak{a}$ we have

$$Z_{\mathfrak{g}}(v) = \mathfrak{g}^0 + \sum_{\substack{\beta \in R^+ \\ \beta(v)=0}} \mathfrak{g}^{\beta} = Z_{\mathfrak{K}}(\mathfrak{a}) + (\mathfrak{a} + \sum_{\substack{\beta \in R^+ \\ \beta(v)=0}} \mathfrak{p}^{\beta})$$

If we consider the geodesic $\gamma(t) = \exp_G(tv) \cdot p_0$, then $K_{\gamma} \equiv \{k \in K : k \circ \gamma = \gamma\}$ is identical to the group $K_{\gamma'(0)} = \{k \in K : k \cdot \gamma'(0) = \gamma'(0)\}$ and has Lie algebra $\mathfrak{K}_v \subseteq \mathfrak{K}$ given by

$$(2.6) \quad \mathfrak{K}_v = \{x \in \mathfrak{K} : [x, v] = 0\} = Z_{\mathfrak{K}}(v) = Z_{\mathfrak{K}}(\mathfrak{a}) + \sum_{\substack{\beta \in R^+ \\ \beta(v)=0}} \mathfrak{K}^{\beta}.$$

Therefore, letting $G \cdot v \simeq G/G_v = G/K_v$ denote the orbit of v , we see that $\dim G \cdot v \leq \dim G - \dim Z_{\mathfrak{K}}(\mathfrak{a})$ with equality if and only if v is a regular vector. In the case where M is a compact Lie group U , we can state this as follows.

Lemma 2.7. *Let (U, g) be a compact Lie group equipped with a bi-invariant metric and $v \in \Lambda_I(U)$ give rise to a closed geodesic of length τ . Then*

$$\dim \text{Fix}_v(\Phi_\tau) \leq 2 \dim U - \text{rank}(U),$$

with equality if and only if v is a regular vector. Here, $\text{Fix}_v(\Phi_\tau) = (U \times U) \cdot \frac{v}{\|v\|}$ denotes the component of $\text{Fix}(\Phi_\tau)$ containing $\frac{v}{\|v\|}$.

We now supply the proof of Theorem 1.16, which states that the rank of a compact Lie group U can be recovered from a generic bi-invariant metric carried by U .

Proof or Theorem 1.16. For any $g \in \mathcal{W}_0(U)$, it follows from Equation 1.5 that $\dim(\text{Fix}(\Phi_\tau))$ is encoded in the spectrum of (U, g) . Also, it follows from the classification of root systems that every integral lattice contains a regular vector (cf. Appendix A). Therefore, by Lemma 2.7 and the fact that the dimension of U is a spectral invariant, we observe that

$$\text{rank}(U) = 2 \dim(U) - \max_{\tau \in \text{Spec}_L(U, g)} \dim(\text{Fix}(\Phi_\tau))$$

can be recovered from the spectrum of (U, g) . The last statement of the proposition follows immediately. \square

Definition 2.8. For any $v \in \mathfrak{a}$, its *degree of singularity* is defined to be the nonnegative integer

$$\begin{aligned} \deg_{\text{sing}}(v) &= \sum_{\substack{\beta > 0 \\ \beta(v)=0}} n_\alpha \\ &= \dim \mathfrak{K}_v - \dim Z_{\mathfrak{K}}(\mathfrak{a}) \\ &= \dim G - \dim Z_{\mathfrak{K}}(\mathfrak{a}) - \dim G \cdot v \\ &= \tilde{C}(M) - \dim G \cdot v, \end{aligned}$$

where $G \cdot v$ denotes the orbit of v under the natural action of G on TM and $\tilde{C}(M) \equiv \dim G - \dim Z_{\mathfrak{K}}(\mathfrak{a})$ is a constant depending only on M .

2.3. The Morse Index of a Closed Geodesic. As we noted in the introduction, in order to understand whether the 0-th wave invariants associated to a clean length τ vanish one needs to be able to compute the Morse index of a closed geodesic. In this subsection we recall a result of Ziller which states that in a compact symmetric space, the Morse index of a closed geodesic can be computed in terms of the roots of the symmetric space.

Given a Riemannian manifold (M, g) we let \mathcal{P} denote the space of piecewise smooth curves $c : [0, T] \rightarrow M$ and we recall that the energy functional $E : \mathcal{P} \rightarrow \mathbb{R}$ is given by

$$E(c) = \int_0^T \|c'(t)\|^2 dt.$$

As usual we consider the critical points of E subject to some boundary conditions. Indeed, for any smooth manifold $B \subset M \times M$ we let $\mathcal{P}_B = \{c \in \mathcal{P} : (c(0), c(T)) \in B\}$. Then $c \in \mathcal{P}_B$ is said to be a *stationary curve* of E restricted to \mathcal{P}_B , if $DE_c(X) = 0$ for any $X \in T_c\mathcal{P}_B$ and its *Morse index* (with respect to the boundary condition B), denoted by $\sigma_B(c)$, is defined to be

$$\sigma_B(c) \equiv \sup\{\dim L : L \leq T_c\mathcal{P}_B \text{ and } D^2E_c \upharpoonright L \text{ is negative definite}\}.$$

In the case where $B = \{(p, q)\}$ for some $p, q \in M$ or $B = \Delta M \equiv \{(p, p) : p \in M\}$, it is known that $\sigma_B(c)$ is finite.

In the introduction we saw that in order to compute the 0-th wave invariant of a clean length τ in the length spectrum of (M, g) , one must be able to compute $\sigma_{\Delta M}(\gamma)$ for any closed geodesic γ of length τ . In general, one has

$$\sigma_{\Delta M}(\gamma) = \sigma_{(\gamma(0), \gamma(T))}(\gamma) + \text{Conv}(\gamma),$$

where $\text{Conv}(\gamma)$ is an integer between 0 and $\dim M - 1$ known as the *concavity* of γ (cf. [BTZ, Eq. 1.4']). However, in the case where our manifold is homogeneous, Ziller has shown that the concavity vanishes [Z1, Theorem 1]. Hence, it follows from the Morse index theorem that we may compute $\sigma_{\Delta M}(\gamma)$ in terms of conjugate points. And, in the case of a *compact* symmetric space this can be realized in terms of the restricted roots of M .

Theorem 2.9 ([Z3], p. 11-12). *With the notation as above, assume that $M = G/K$ is a compact symmetric space. Now, let $v \in \mathfrak{a} \subset \mathfrak{p}$ be such that $\gamma(t) = \exp_G(tv) \cdot p_0$, $0 \leq t \leq 1$, is a closed geodesic. Then the Morse index of γ in the space of closed geodesics in M is given by*

$$(2.10) \quad \sigma_{\Delta M}(\gamma) = \sum_{\beta \in R^+ \cup \{0\}} n_\beta |\beta(v)| - \sum_{\substack{\beta \in R^+ \\ \beta(v) \neq 0}} n_\beta.$$

If γ_v in the above is a closed geodesic of length $\tau = \|v\|$, then in terms of the degree of singularity, Equation 2.10 becomes $\sigma_{\Delta M}(\gamma_v) = \sum_{\beta \in R^+} n_\beta |\beta(v)| - \dim(M) + \text{rank}(M) + \deg_{\text{sing}}(v)$, and, since $\mathfrak{K} = Z_{\mathfrak{K}}(\mathfrak{a}) \oplus (\oplus_{\beta \in R^+} \mathfrak{K}^\beta)$, it follows that

$$(2.11) \quad \sigma_{\Delta M}(\gamma_v) = \begin{cases} 0 & \text{for } v = 0 \\ \sum_{\beta \in R^+} n_\beta |\beta(v)| - \dim \text{Fix}_v(\Phi_\tau) + \dim M & \text{for } v \neq 0, \end{cases}$$

where $\text{Fix}_v(\Phi_\tau) \equiv G \cdot \frac{v}{\|v\|}$ is the component of $\text{Fix}(\Phi_\tau) \subseteq SM$ containing $\frac{v}{\|v\|}$. We will say that a vector $v \in \mathfrak{a}$ is *Euclidean* if $\beta(v) = 0$ for all $\beta \in R^+$ or, equivalently, $G \cdot v$ is diffeomorphic to M . Since $\text{Fix}_v(\Phi_\tau)$ is diffeomorphic to $G \cdot v$ for $v \neq 0$, we conclude that when v is Euclidean $\sigma_{\Delta M}(\gamma_v)$ vanishes, as expected. Now, as we noted in Section 2.1, all geodesics in M can be conjugated via the isometry group to a geodesic having initial conditions in \mathfrak{a} . Therefore, we have an effective means for computing the Morse Index of any closed geodesic in a compact symmetric space.

2.4. The Center of a Symmetric Space and Some Useful Lattices. We recall that an isometry f of a Riemannian manifold is said to be a *transvection* if there is a point q at which df is equal to the parallel transport along a piecewise smooth curve joining q to $f(q)$. If we let $\mathcal{T}(M)$ denote the group of transvections of a symmetric space M , then the *center* of M , denoted by $Z(M)$, is defined to be the centralizer of $\mathcal{T}(M)$ in $\text{Isom}(M)$. We then have the following well-known result concerning Riemannian coverings.

Proposition 2.12 ([Wo], Theorem 8.3.11). *Let M be a symmetric space with center $Z(M)$.*

- (1) *Let $\pi : M \rightarrow N$ be a Riemannian covering with deck transformation group Γ . If N is a symmetric space, then Γ is a discrete subgroup of $Z(M)$.*
- (2) *If $\Gamma \leq Z(M)$ is a discrete subgroup, then $N = \Gamma \backslash M$ is a symmetric space and the natural projection $\pi : M \rightarrow \Gamma \backslash M$ is a Riemannian covering with deck transformation group Γ .*

Consequently, if \widetilde{M} is a simply-connected symmetric space, then any symmetric space covered by \widetilde{M} is of the form $\Gamma \backslash \widetilde{M}$ for some discrete subgroup of $Z(\widetilde{M})$.

Let $R = R(M)$ be the restricted roots of a symmetric space M with respect to some maximal abelian subspace $\mathfrak{a} \subset \mathfrak{p} \equiv T_{p_0}M$. Then for each $\alpha \in R$ we define its associated *co-root* α^\vee to be the vector in \mathfrak{a} satisfying

- (1) α^\vee is orthogonal to $\ker \alpha$,
- (2) $\alpha(\alpha^\vee) = 2$.

The co-root, central and integral lattices of M are defined by

$$\begin{aligned} \Lambda_R(M) &= \langle \alpha^\vee : \alpha \in R \rangle \\ \Lambda_Z(M) &= \{v \in \mathfrak{a} : \alpha(v) \in \mathbb{Z} \text{ for all } \alpha \in R\} \\ \Lambda_I(M) &= \{v \in \mathfrak{a} : \exp_G(v) \cdot p_0 = p_0\}. \end{aligned}$$

Clearly, if M_1 and M_2 are symmetric spaces, then $\Lambda_Y(M_1 \times M_2) = \Lambda_Y(M_1) \times \Lambda_Y(M_2)$, where $Y = R^\vee, Z, I$.

Proposition 2.13 (cf. [Lo] Theorems VI.2.4 and VI.3.6). *Let $M = \Gamma \backslash (M_0 \times \widetilde{M}_1 \times \cdots \times \widetilde{M}_q)$ be the homogeneity type of a compact symmetric space and set $d = \dim M_0$. Then the following hold.*

- (1) $\Lambda_I(M_0 \times \widetilde{M}_{\text{cpt}}) \simeq \mathbb{Z}^d \times \Lambda_I(\widetilde{M}_{\text{cpt}}) = \mathbb{Z}^d \times \Lambda_I(\widetilde{M}_1) \times \cdots \times \Lambda_I(\widetilde{M}_q)$
- (2) $\Lambda_R(M) = \Lambda_R(M_0 \times \widetilde{M}_{\text{cpt}}) = 0 \times \Lambda_R(\widetilde{M}_1) \times \cdots \times \Lambda_R(\widetilde{M}_q)$.
- (3) $\Lambda_Z(M) = \Lambda_Z(M_0 \times \widetilde{M}_{\text{cpt}}) = \mathbb{R}^d \times \Lambda_Z(\widetilde{M}_1) \times \cdots \times \Lambda_Z(\widetilde{M}_q)$.
- (4) $\Lambda_R(M) \leq \Lambda_I(M) \leq \Lambda_Z(M)$.
- (5) $Z(M) \simeq \Lambda_Z(M) / \Lambda_I(M)$
- (6) *Let $\pi : \Lambda_Z(M_0 \times \widetilde{M}_{\text{cpt}}) \rightarrow Z(M_0 \times \widetilde{M}_{\text{cpt}}) \equiv \Lambda_Z(M_0 \times \widetilde{M}_{\text{cpt}}) / \Lambda_I(M_0 \times \widetilde{M}_{\text{cpt}})$ be the natural projection, then*

$$\Lambda_I(M) = \pi^{-1}(\Gamma)$$

$$(7) \quad \pi_1(M) \simeq \Lambda_I(M)/\Lambda_R(M); \text{ in particular, } \Lambda_I(\widetilde{M}_{\text{cpt}}) = \Lambda_R(\widetilde{M}_{\text{cpt}}).$$

3. HEARING THE LENGTH SPECTRUM OF A SPLIT-RANK SYMMETRIC SPACE

In this section we complete the proof of Theorem 1.13 by proving Proposition 1.18, Lemma 1.22 and Theorem 1.23. We also provide an infinite family of examples demonstrating that Theorem 1.23 cannot hold for an arbitrary (irreducible) symmetric space, which suggests that exploring the conjecture through the trace formula of Duistermaat and Guillemin will prove to be more delicate in general.

3.1. Component Lengths and the Morse Index. In this section we provide the argument in support of Proposition 1.18 which tells us that in a compact symmetric space M for which the non-Euclidean part of its universal cover is split-rank, the Morse index modulo 4 of a closed geodesic γ of length τ is determined by the length of the components of $\gamma'(t)$ and the dimension of $\text{Fix}_\gamma(\Phi_\tau)$, the component of $\text{Fix}(\Phi_\tau)$ corresponding to γ .

Proof of Proposition 1.18. Let $M = \Gamma \backslash (M_0 \times \widetilde{M}_1 \times \cdots \times \widetilde{M}_q)$ be the homogeneity type of a compact symmetric space, where $\widetilde{M}_1 \times \cdots \times \widetilde{M}_q$ is split-rank or, equivalently, \widetilde{M}_j is split rank for $j = 1, \dots, q$, and let g be the symmetric metric on M induced by the symmetric metric $h \times g_1 \times \cdots \times g_q$ on $M_0 \times \widetilde{M}_1 \times \cdots \times \widetilde{M}_q$. Furthermore, for $j = 0, 1, \dots, q$, Γ_j will denote the projection of $\Gamma \leq Z(M_0 \times \widetilde{M}_1 \times \cdots \times \widetilde{M}_q) = M_0 \times Z(\widetilde{M}_1) \times \cdots \times Z(\widetilde{M}_q)$ onto the j -th factor and $M_j \equiv \Gamma_j \backslash \widetilde{M}_j$.

Now, for each $j = 1, \dots, q$, let R_j denote the restricted root system of $\widetilde{M}_j = G_j/K_j$ with respect to some choice of maximal abelian subspace $\mathfrak{a}_j \subset \mathfrak{p}_j \equiv T_o \widetilde{M}_j$ and let R_j^+ denote the positive roots with respect to some choice of Weyl chamber $\mathcal{C}_j \subset \mathfrak{a}_j$. Then, letting \mathfrak{a}_0 denote the Lie algebra of the torus M_0 , the restricted root system for M with respect to the maximal abelian subspace $\mathfrak{a} = \mathfrak{a}_0 \times \mathfrak{a}_1 \times \cdots \times \mathfrak{a}_q \subset \mathfrak{p} \equiv T_o M$ is given by

$$R = \{\widetilde{\beta}_j = \beta_j \circ \pi_j : \beta_j \in R_j \text{ for } j = 1, \dots, q\},$$

where $\pi_j : \mathfrak{a}_1 \times \cdots \times \mathfrak{a}_q \rightarrow \mathfrak{a}_j$ is given by $v = (v_1, \dots, v_q) \mapsto v_j$, and the positive roots with respect to the Weyl chamber $\mathcal{C} = \bigoplus_{j=1}^q \mathcal{C}_j$ are given by

$$R^+ = \{\widetilde{\beta}_j = \beta_j \circ \pi_j : \beta_j \in R_j^+ \text{ for } j = 1, \dots, q\}.$$

It follows from Equation 2.11 that for any $v = (v_0, v_1, \dots, v_q) \neq 0 \in \overline{\mathcal{C}} \cap \Lambda_I(M) \subset \overline{\mathcal{C}} \cap \Lambda_Z(M)$ the Morse index of the associated closed geodesic γ_v is given by

$$\begin{aligned} (3.1) \quad \sigma_{\Delta M}(\gamma_v) &= \sum_{\beta \in R^+} n_\beta \beta(v) - \dim(\text{Fix}_{\gamma_v}(\Phi_\tau)) + C(M) \\ &= \sum_{j=1}^q \sum_{\beta_j \in R_j^+} n_{\beta_j} \beta_j(v_j) - \dim(\text{Fix}_{\gamma_v}(\Phi_\tau)) + C(M), \end{aligned}$$

where, as in Section 2, n_β and n_{β_j} denote the multiplicities of the roots. Therefore, each non-Euclidean factor \widetilde{M}_j of the universal cover of M makes a contribution of $\sum_{\beta_j \in R_j^+} n_{\beta_j} \beta_j(v_j)$ to the Morse index of γ_v . We will show that for each $j = 1, \dots, q$ there is a function $f_{(M_j, g_j)}$ such that $\sum_{\beta_j \in R_j^+} n_{\beta_j} \beta_j(v_j) \equiv f_{(M_j, g_j)}(\|v_j\|) \pmod{4}$, as v_j ranges over $\Lambda_I(M_j)$. Then, since $\Lambda_I(M) \leq \Lambda_I(M_0) \times \Lambda_I(M_1) \times \dots \times \Lambda_I(M_q)$, the theorem follows by taking $f_{(M, g)} \equiv \sum_{j=1}^q f_{(M_j, g_j)}$ and recalling that every closed geodesic is conjugate via the isometry group to a closed geodesic in our chosen maximal flat.

In light of the previous discussion, henceforth, we will assume that (M, g) is an irreducible split-rank symmetric space and we set out to prove there is a function $f_{(M, g)}$ such that

$$(3.2) \quad \sum_{\beta \in R^+} n_\beta \beta(v) \equiv f_{(\widetilde{M}, g)}(\|v\|) \pmod{4},$$

for any $v \in \Lambda_I(M) \cap \overline{\mathcal{C}}$. At times the following observation will be useful.

Lemma 3.3. *Let $v = (c_1, \dots, c_n), w = (d_1, \dots, d_n) \neq 0 \in \mathbb{Z}^n$ be such that $\sum_{j=1}^n c_j^2 = \sum_{j=1}^n d_j^2 = \tau^2$, then $\sum_{j=1}^n c_j$ and $\sum_{j=1}^n d_j$ are congruent modulo 2. That is, $\|v\|^2 = \|w\|^2$, implies the number of odd c_n 's is even if and only if the number of odd d_n 's is even.*

Proof. To the contrary, let's assume that $\sum_{j=1}^n c_j = 2k+1$ is odd and $\sum_{j=1}^n d_j = 2m$ is even. Then $(2k+1)^2 = (\sum_{j=1}^n c_j)^2 = \tau^2 + 2 \sum_{i < j} c_i c_j$ and $(2m)^2 = (\sum_{j=1}^n d_j)^2 = \tau^2 + 2 \sum_{i < j} d_i d_j$. This implies that $(2k+1)^2 - (2m)^2$ is even, which is a contradiction. \square

Lemma 3.4. *Let (M, g) be an irreducible simply-connected split rank symmetric space. Then $f_{(M, g)} \equiv 0$.*

Proof. Since (M, g) is an irreducible split-rank space we know there is an integer N such that for each $\beta \in R^+$, we have $n_\beta = 2N$ with $N = 1$ if and only if M is a simple Lie group [Lo, Theorems VI.4.3 and VI.4.4]. (In fact, from the classification of irreducible symmetric spaces, one can see that $2N$ equals 2, 4 or 8.) Since M is simply-connected, we know that the co-root lattice and integral lattice coincide: $\Lambda_R(M) = \Lambda_I(M)$. Therefore, since the lowest strongly dominant form $\rho = \frac{1}{2} \sum_{\beta \in R^+} \beta$ is integer-valued on the co-root lattice (see Lemma A.5), we see that for any $v \in \overline{\mathcal{C}} \cap \Lambda_I(M)$, we have $\sum_{\beta \in R^+} n_\beta \beta(v) = 4N\rho(v) \equiv 0 \pmod{4}$. Therefore, we may take $f_{(M, g)}$ to be identically zero, which concludes the proof of the lemma. \square

Lemma 3.5. *Let (M, g) be an irreducible split-rank symmetric space that is not a simple Lie group; that is, (M, g) is an irreducible split-rank symmetric space of Type I. Then $f_{(M, g)} \equiv 0$ satisfies Equation 3.2.*

Proof. As we noted in the proof of Lemma 3.4, since (M, g) is split-rank and not a simple Lie group, there is an integer $N \geq 2$, such that for each $\beta \in R^+$, we have $n_\beta = 2N$. Therefore, since the roots are integer-valued on any $v \in \overline{\mathcal{C}} \cap \Lambda_Z(\widetilde{M})$ we conclude that when (M, g) is Type I, the quantity $\sum_{\beta \in R^+} n_\beta \beta(v)$ is divisible by four and we may take $f_{(M, g)} \equiv 0$. \square

We now consider the case where (M, g) is a simple Lie group U equipped with a bi-invariant metric. As we noted in the proof of Lemma 3.4, $n_\beta = 2$ for each $\beta \in R^+$. By examining each type of simple Lie group we will produce a function $f_{(M, g)}$ such that

$$\sum_{\beta \in R^+} n_\beta \beta(v) = 4\rho(v) \equiv f_{(M, g)}(\|v\|) \pmod{4}.$$

In many of the cases considered this function will be identically zero.

Standing Assumption: Letting $B : \mathfrak{u} \times \mathfrak{u} \rightarrow \mathbb{R}$ be the Killing form on the Lie algebra \mathfrak{u} of U , for the remainder of the argument we will let \bar{g} be the metric on $M = U \times U/\Delta U$ induced by $-r(B \oplus B)$, where $r > 0$ is chosen to agree with the inner-product structure in Section A.2. With this choice of r , \bar{g} is the bi-invariant metric on $M \simeq U$ induced by $-2rB$ on \mathfrak{u} . Then, the metric g equals $c\bar{g}$ for some $c > 0$.

Lemma 3.6. *Let (M, g) be a simple group of type A_n ($n \geq 1$) equipped with a bi-invariant metric g . That is, $M = \mathrm{SU}(n+1)/\Gamma$, where $\Gamma \leq \mathbb{Z}_{n+1}$, with $g = c\bar{g}$ for some $c > 0$.*

- (1) *If n is even or $\Gamma \neq \mathbb{Z}_{n+1}$, then $f_{(M, g)} \equiv 0$ satisfies Equation 3.2.*
- (2) *If n is odd and $\Gamma = \mathbb{Z}_{n+1}$, then*

$$f_{(M, g)}(\|v\|) = \begin{cases} 0 & \text{when } (n+1) \frac{\|v\|^2}{c} \text{ is an even integer} \\ 2 & \text{when } (n+1) \frac{\|v\|^2}{c} \text{ is an odd integer.} \end{cases}$$

satisfies Equation 3.2.

Proof. Using the notation of Section A.2.1, we recall that the central lattice is given by $\Lambda_Z = \langle L_j = \frac{n}{n+1}e_j - \frac{1}{n+1} \sum_{\substack{k=1 \\ k \neq j}}^{n+1} e_k : j = 1, \dots, n \rangle$. Now, let $v = \sum_{j=1}^n k_j L_j \in \Lambda_I \cap \bar{\mathcal{C}} \subseteq \Lambda_Z \cap \bar{\mathcal{C}}$, then

$$\begin{aligned} 2 \sum_{\alpha \in R^+} \alpha(v) &= \sum_{j=1}^n 2k_j \sum_{\alpha \in \Delta^+} \alpha(L_j) \\ &= \sum_{j=1}^n 2k_j \sum_{\mu=1}^n 2(n-\mu+1) \epsilon_\mu(L_j) \\ &= \sum_{j=1}^n 4k_j \sum_{\mu=1}^n (n-\mu+1) \epsilon_\mu(L_j) \\ &= \sum_{j=1}^n 4k_j \left((n-j+1) \frac{n}{n+1} - \frac{1}{n+1} \sum_{\substack{\mu=1 \\ \mu \neq j}}^n (n-\mu+1) \right) \\ &= \sum_{j=1}^n 4k_j \left((n-j+1) \frac{n}{n+1} - \frac{1}{n+1} \left(\frac{n(n+1)}{2} - (n-j+1) \right) \right) \\ &= \sum_{j=1}^n 4k_j \left((n-j+1) - \frac{n}{2} \right) \end{aligned}$$

$$\begin{aligned}
&= \sum_{j=1}^n 4k_j \left(\frac{n-2j+2}{2} \right) \\
&= \sum_{j=1}^n 2k_j(n-2j+2) \\
&\equiv 2n \left(\sum_{j=1}^n k_j \right) \pmod{4}
\end{aligned}$$

Therefore, in the case where n is even, this quantity is always congruent to zero modulo 4, so we may take $f_{(M,g)} \equiv 0$.

In the case that n is odd, we see that $2 \sum_{\alpha \in \Delta^+} \alpha(v)$ is congruent to 0 (resp. 2) modulo 4 if the number of k_j 's that are odd is even (resp. odd). However, the parity of the number of odd k_j 's is determined by the length of the vector v . Indeed, $\|v\|^2 = \frac{n}{n+1} \sum_{j=1}^n k_j^2 - \frac{2}{n+1} \sum_{i < j} k_i k_j$, so that $(n+1)\|v\|^2$ is an integer and this integer is even if and only if the number of k_j 's that are odd is even. To see which quotients $\text{SU}(2n+1)/\Gamma$ admit $v \in \Lambda_I(\text{SU}(2n+1)/\Gamma)$ with $(n+1)\|v\|^2$ odd, recall that Λ_Z is also generated by $\langle L_1, e_\mu - e_\nu : 1 \leq \mu < \nu \leq n+1 \rangle$ (See Section A.2.1). Therefore, we may express $v \in \Lambda_I \cap \bar{\mathcal{C}}$ as $v = \hat{k}_1 L_1 + \sum_{i < j} c_{ij}(e_i - e_j) = \hat{k}_1 L_1 + \sum_{j=1}^{n+1} C_j e_j$. It follows that when n is odd, $(n+1)\|v\|^2$ is an odd integer if and only if \hat{k}_1 is odd. By Equation A.6 the integral lattice of $M = \text{SU}(n+1)/\Gamma$ is $\Lambda_I(M) = \langle kL_1 \rangle + \Lambda_R$, where $k = 0, 1, \dots, n$ is the smallest generator of $\Gamma \leq \mathbb{Z}_{n+1}$. Therefore, in the event that n is odd, \hat{k}_1 can be odd only when $\Gamma = \mathbb{Z}_{n+1}$. And, the lemma follows. \square

Lemma 3.7. *Let (M, g) be a simple group of type B_n ($n \geq 2$) equipped with a bi-invariant metric g . That is, $M = \text{Spin}(2n+1)/\Gamma$, where $\Gamma \leq Z(\text{Spin}(2n+1)) \simeq \mathbb{Z}_2$, with $g = c\bar{g}$ for some $c > 0$.*

- (1) *If Γ is trivial, then $f_{(M,g)} \equiv 0$ satisfies Equation 3.2.*
- (2) *If $\Gamma = \mathbb{Z}_2$, then*

$$f_{(M,g)}(\|v\|) = \begin{cases} 0 & \text{when } \frac{\|v\|^2}{c} \text{ is an even integer} \\ 2 & \text{when } \frac{\|v\|^2}{c} \text{ is an odd integer.} \end{cases}$$

satisfies Equation 3.2.

Proof. The first statement follows from Lemma 3.4. Now, assume that $\Gamma = \mathbb{Z}_2$, so that the integral lattice $\Lambda_I(M)$ equals the central lattice Λ_Z . As is noted in Section A.2.2 the central lattice is given by $\Lambda_Z = \langle e_1, e_1 - e_2, \dots, e_1 - e_n \rangle = \langle e_1, \dots, e_n \rangle$ and the sum of the positive roots is $\sum_{\alpha \in R^+} \alpha = \sum_{j=1}^n (2n-2j+1)\epsilon_j$. Then, for $v = \sum_{j=1}^n k_j e_j \in \Lambda_I \cap \bar{\mathcal{C}} = \Lambda_Z \cap \bar{\mathcal{C}}$ we have

$$\begin{aligned}
2 \sum_{\alpha \in R^+} \alpha(v) &= \sum_{j=1}^n 4(n-j)\epsilon_j(v) + 2 \sum_{j=1}^n \epsilon_j(v) \\
&\equiv 2 \sum_{j=1}^n \epsilon_j(v) \pmod{4} \\
&= 2 \sum_{j=1}^n k_j \pmod{4}.
\end{aligned}$$

Then, by Lemma 3.3, if $w = \sum_{j=1}^n \hat{k}_j e_j \in \Lambda_I \cap \bar{\mathcal{C}}$ is such that $\frac{\|w\|^2}{c} = \frac{\|v\|^2}{c}$, we have $\sum_{j=1}^n k_j$ is congruent to $\sum_{j=1}^n \hat{k}_j$ modulo 2, which implies that

$$2 \sum_{j=1}^n k_j \equiv 2 \sum_{j=1}^n \hat{k}_j \pmod{4}.$$

In the case where $\frac{\|w\|^2}{c} = \frac{\|v\|^2}{c}$ is an even integer, this quantity is zero modulo 4. In the case where $\frac{\|w\|^2}{c} = \frac{\|v\|^2}{c}$ is odd, this quantity is 2 modulo 4. \square

Lemma 3.8. *Let (M, g) be a simple group of type C_n ($n \geq 3$) equipped with a bi-invariant metric g . That is, $M = \mathrm{Sp}(n)/\Gamma$, where $\Gamma \leq Z(\mathrm{Sp}(n)) \simeq \mathbb{Z}_2$, with $g = c\bar{g}$ for some $c > 0$.*

- (1) *If $n \equiv 0, 3 \pmod{4}$ or Γ is trivial, then $f_{(M,g)} \equiv 0$ satisfies Equation 3.2.*
- (2) *If $n \equiv 1, 2 \pmod{4}$ and $\Gamma = Z(\mathrm{Sp}(n)) \simeq \mathbb{Z}_2$, then*

$$f_{(M,g)}(\|v\|) = \begin{cases} 0 & \text{when } \frac{\|v\|^2}{c} \text{ is an integer} \\ 2 & \text{when } \frac{\|v\|^2}{c} \text{ is not an integer.} \end{cases}$$

satisfies Equation 3.2.

Proof. If we let $F = \frac{1}{2} \sum_{j=1}^n e_j$, then according to Section A.2.3 the central lattice is given by $\Lambda_Z = \langle e_1, \dots, e_{n-1}, F \rangle$ and the sum of the positive roots is $\sum_{\alpha \in R^+} \alpha = \sum_{j=1}^n 2(n-j+1)\epsilon_j$. Now, let $v = \sum_{j=1}^{n-1} k_j e_j + k_n F \in \Lambda_I \cap \bar{\mathcal{C}} \subseteq \Lambda_Z \cap \bar{\mathcal{C}}$, then we have

$$\begin{aligned}
2 \sum_{\alpha \in R^+} \alpha(v) &= \sum_{j=1}^n 4(n-j+1)\epsilon_j(v) \\
&= \sum_{j=1}^{n-1} 4(n-j+1)k_j + k_n \sum_{j=1}^n 4(n-j+1)\epsilon_j(F) \\
&\equiv 4k_n \sum_{j=1}^n (n-j+1)\epsilon_j(F) \pmod{4} \\
&\equiv 4k_n \sum_{j=1}^n (n-j+1) \frac{1}{2} \pmod{4}
\end{aligned}$$

$$\begin{aligned}
&\equiv 2k_n \sum_{j=1}^n (n-j+1) \pmod{4} \\
&\equiv 2k_n \left(\frac{n(n+1)}{2} \right) \\
&\equiv k_n n(n+1) \\
&\equiv \begin{cases} 0 \pmod{4}, & \text{for } k_n \text{ even} \\ 0 \pmod{4}, & \text{for } k_n \text{ odd and } n \equiv 0, 3 \pmod{4} \\ 2 \pmod{4}, & \text{for } k_n \text{ odd and } n \equiv 1, 2 \pmod{4}. \end{cases}
\end{aligned}$$

The calculation above and Lemma 3.4 allow us to deduce that for $n \equiv 0, 3 \pmod{4}$ or Γ trivial, we can take $f_{(M,g)}(\|v\|) \equiv 0$.

Now, for any $v = \sum_{j=1}^{n-1} k_j e_j + k_n F \in \Lambda_Z \cap \overline{\mathcal{C}}$, we see that $\|v\|^2/c$ is an integer if and only if k_n is even. This observation plus the calculation above establishes the second statement of the lemma. \square

Lemma 3.9. *Let (M, g) be a simple group of type D_n ($n \geq 4$) equipped with a bi-invariant metric g . That is, $M = \text{Spin}(2n)/\Gamma$, where*

$$\Gamma \leq Z(\text{Sp}(n)) \simeq \begin{cases} \mathbb{Z}_4 & n \text{ even} \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2 & n \text{ odd}, \end{cases}$$

with $g = c\bar{g}$ for some $c > 0$. Then Equation 3.2 has a solution $f_{(M,g)}$ having the following form.

- (1) If $n \equiv 0, 1 \pmod{4}$, then $f_{(M,g)} \equiv 0$.
- (2) Let $n \equiv 3 \pmod{4}$.
 - (a) If Γ trivial or $\mathbb{Z}_2 \oplus 1$, then $f_{(M,g)} \equiv 0$;
 - (b) If Γ is $1 \oplus \mathbb{Z}_2$, $\langle (1, 1) \rangle \simeq \mathbb{Z}_2$, or $Z(\text{Spin}(2n)) \simeq \mathbb{Z}_2 \oplus \mathbb{Z}_2$, then

$$f_{(M,g)}(\|v\|) = \begin{cases} 0 & \text{when } \frac{\|v\|^2}{\frac{c}{c}} \text{ is an integer} \\ 2 & \text{when } \frac{\|v\|^2}{\frac{c}{c}} \text{ is not an integer.} \end{cases}$$

- (3) Let $n \equiv 2 \pmod{4}$.
 - (a) If Γ is trivial or \mathbb{Z}_2 , then $f_{(M,g)} \equiv 0$;
 - (b) If $\Gamma = Z(\text{Spin}(2n)) \simeq \mathbb{Z}_4$, then

$$f_{(M,g)}(\|v\|) = \begin{cases} 0 & \text{when } \frac{\|v\|^2}{\frac{c}{c}} \text{ is an integer} \\ 2 & \text{when } \frac{\|v\|^2}{\frac{c}{c}} \text{ is not an integer.} \end{cases}$$

Proof. As is noted in Section A.2.4, the central lattice is given by $\Lambda_Z = \langle e_1, \dots, e_{n-1}, F \rangle$, where $F = \frac{1}{2} \sum_{j=1}^n e_j$, and the sum of the positive roots is $\sum_{\alpha \in R^+} \alpha = \sum_{j=1}^n 2(n-j)\epsilon_j =$

$\sum_{j=1}^{n-1} 2(n-j)\epsilon_j$. Now, let $v = \sum_{j=1}^{n-1} k_j e_j + k_n F \in \Lambda_I \cap \overline{\mathcal{C}} \subseteq \Lambda_Z \cap \overline{\mathcal{C}}$, then we have

$$\begin{aligned}
2 \sum_{\alpha \in R^+} \alpha(v) &= \sum_{j=1}^n 4(n-j)\epsilon_j(v) \\
&= \sum_{j=1}^{n-1} 4k_j(n-j) + k_n \sum_{j=1}^n 4(n-j)\epsilon_j(F) \\
&\equiv 2k_n \sum_{j=1}^n (n-j) \pmod{4} \\
&\equiv 2k_n \left(\frac{n(n-1)}{2} \right) \\
&\equiv k_n n(n-1) \\
&\equiv \begin{cases} 0 \pmod{4}, & \text{for } k_n \text{ even} \\ 0 \pmod{4}, & \text{for } k_n \text{ odd and } n \equiv 0, 1 \pmod{4} \\ 2 \pmod{4}, & \text{for } k_n \text{ odd and } n \equiv 2, 3 \pmod{4}. \end{cases}
\end{aligned}$$

Consulting the Equations A.9 and A.10, we see that statements (1), (2a) and (3a) of the lemma follow from the computation above. Now, notice that the parity of k_n is determined by $\|v\|$ and $\|v\|^2/c$ is an integer if and only if k_n is even. Then, statements (2b) and (3b) follow from the computation above and Equations A.9 and A.10. \square

Lemma 3.10. *Let (M, g) be the simple Lie group of Type F_4 equipped with a bi-invariant metric g , then $f_{(M, g)} \equiv 0$ satisfies Equation 3.2.*

Proof. Since M is simply-connected the result follows from Lemma 3.4. \square

Lemma 3.11. *Let (M, g) be the simple Lie group of Type G_2 equipped with a bi-invariant metric g , then $f_{(M, g)} \equiv 0$ satisfies Equation 3.2.*

Proof. Since M is simply-connected the result follows from Lemma 3.4. \square

Lemma 3.12. *Let (M, g) be the simple Lie group of Type E_8 equipped with a bi-invariant metric g , then $f_{(M, g)} \equiv 0$ satisfies Equation 3.2.*

Proof. Since M is simply-connected the result follows from Lemma 3.4. Alternatively, noting that $D_8 \subset E_8$ implies that $\Lambda_Z^{E_8} \subset \Lambda_Z^{D_8}$, this lemma follows from Lemma 3.9 in the case where $n = 4$. \square

Lemma 3.13. *Let (M, g) be a simple Lie group of type E_7 equipped with a bi-invariant metric g . That is, letting \tilde{E}_7 denote the unique compact simply-connected Lie group of type E_7 , $M = \tilde{E}_7/\Gamma$, where $\Gamma \leq Z(\tilde{E}_7) \simeq \mathbb{Z}_2$, with $g = c\tilde{g}$ for some $c > 0$.*

(1) *If Γ is trivial, then $f_{(M, g)} \equiv 0$ satisfies Equation 3.2.*

(2) If $\Gamma = Z(\tilde{E}_7) \simeq \mathbb{Z}_2$, then

$$f_{(M,g)} = \begin{cases} 0 & \frac{\|v\|^2}{c} \text{ is an integer} \\ 2 & \frac{\|v\|^2}{c} \text{ is not an integer} \end{cases}$$

satisfies Equation 3.2.

Proof. The first statement follows from Lemma 3.4. So, we may assume that $M = \tilde{E}_7/Z(\tilde{E}_7)$, in which case the integral lattice $\Lambda_I(M)$ coincides with the central lattice Λ_Z .

Now, let $v \in \Lambda_I \cap \bar{\mathcal{C}} = \Lambda_Z \cap \bar{\mathcal{C}}$, then following the notation and conventions of Section A.2.9, one can see that $v = \frac{1}{2} \sum_{j=1}^6 c_j e_j + \frac{1}{2} c_7 (e_7 - e_8)$ for $c_1, \dots, c_6, c_7 \in \mathbb{Z}$ satisfying $c_1 \equiv \dots \equiv c_6 \pmod{2}$ and c_7 arbitrary. Then we have

$$\begin{aligned} 2 \sum_{\alpha \in R^+} \alpha(v) &= 4\rho(v) \\ &= 34c_7 + 2 \sum_{j=1}^5 (6-j)c_j \\ &\equiv 2c_7 + 2(5c_1 + 4c_2 + 3c_3 + 2c_4 + c_5) \pmod{4} \\ &\equiv \begin{cases} 2c_7 \pmod{4}, & \text{for } c_1, \dots, c_6 \text{ even} \\ 2c_7 + 2 \pmod{4}, & \text{for } c_1, \dots, c_6 \text{ odd.} \end{cases} \end{aligned}$$

Recalling that c_1, \dots, c_6 have the same parity the previous equation becomes

$$2 \sum_{\alpha \in R^+} \alpha(v) = \begin{cases} 0 \pmod{4}, & \text{for } c_7 \equiv c_1 \pmod{2} \\ 2 \pmod{4}, & \text{for } c_7 \not\equiv c_1 \pmod{2}. \end{cases}$$

However, it is easy to deduce that $\|v\|^2/c = \frac{1}{4}(\sum_{j=1}^6 c_j^2 + 2c_7^2)$ is an integer if and only if $c_7 \equiv c_1 \pmod{2}$. And the result follows. \square

Lemma 3.14. *Let (M, g) be a simple Lie group of Type E_6 equipped with a bi-invariant metric g . That is, letting \tilde{E}_6 denote the unique simply-connected compact Lie group of type E_6 , $M = \tilde{E}_6/\Gamma$, where $\Gamma \leq Z(\tilde{E}_6) \simeq \mathbb{Z}_3$, with $g = c\bar{g}$ for some $c > 0$. Then $f_{(M,g)} \equiv 0$ satisfies Equation 3.2.*

Proof. Following the notation of Section A.2.10, we recall that the central lattice of E_6 is generated by the vectors v_1, \dots, v_6 and $F = \frac{2}{3}(e_6 + e_7 - e_8)$. Now, since $v_1, \dots, v_6 \in \Lambda_{R^-}$, Lemma A.5 tells us that $2 \sum_{\alpha \in R^+} \alpha(v_j) = 4\rho(v_j) \equiv 0 \pmod{4}$ for all $j = 1, \dots, 6$. And, since $2 \sum_{\alpha \in R^+} \alpha(F) = 32 \equiv 0 \pmod{4}$, we find that $2 \sum_{\alpha \in R^+} \alpha$ is congruent to zero on the entire central lattice and the result follows. \square

This concludes the proof of the proposition. \square

3.2. CLU Length Spectra are Generic among Symmetric Spaces. In this section we prove Lemma 1.22 which states that a generic compact symmetric space has CLU length spectrum.

Proof of Lemma 1.22. Let $\Lambda_I(M) \leq \mathfrak{a} = \mathfrak{a}_0 \times \mathfrak{a}_1 \times \cdots \times \mathfrak{a}_q \subseteq \mathfrak{p}$ denote the integral lattice of M and let h_A denote the flat metric on M_0 corresponding to the positive definite real symmetric matrix $A \in \mathcal{S}^+(d)$. If g is the symmetric metric on M induced by the metric $h_A \times c_1^2 g_0^1 \times \cdots \times c_q^2 g_0^q$ on $M_0 \times \widetilde{M}_1 \times \cdots \times \widetilde{M}_q$, then for each $v = (v_0, v_1, \dots, v_q) \in \Lambda_I(M)$ the corresponding closed geodesic $\gamma_v : [0, 1] \rightarrow (M, g)$ has length

$$\tau = (v_0^t A v_0 + \sum_{j=1}^q c_j^2 \|v_j\|_{j,0}^2)^{1/2},$$

where $\|\cdot\|_{j,0}$ is the norm associated with $g_{j,0}^j$, for each $j = 1, \dots, q$. Then for any $v, w \in \Lambda_I(M)$ define the continuous map $Y_{(v,w)} : \mathcal{R}_{\text{symm}}(M) \equiv \mathcal{S}^+(d) \times \mathbb{R}_+ \times \cdots \times \mathbb{R}_+ \rightarrow \mathbb{R}$ via

$$(A, c_1, \dots, c_q) \mapsto v_0^t A v_0 - w_0^t A w_0 + \sum_{j=1}^q c_j^2 (\|v_j\|_{j,0}^2 - \|w_j\|_{j,0}^2).$$

Now, let $\Delta = \{(v, w) \in \Lambda_I(M) \times \Lambda_I(M) : \|v_j\|_{j,0} \neq \|w_j\|_{j,0} \text{ for some } j \geq 1\}$. Then it is clear that a metric g has CLU length spectrum if and only if $g \in \cap_{\delta \in \Delta} Y_\delta^{-1}(\mathbb{R} - \{0\})$. \square

3.3. Determining the Morse Index through Length and Dimension. We present the proof of Theorem 1.23 which establishes that for certain compact symmetric spaces (M, g) for which the non-Euclidean part of its cover is trivial or split-rank, the Morse index modulo 4 of a closed geodesic γ of length τ is determined by τ and $\dim \text{Fix}_\gamma(\Phi_\tau)$.

Proof of Theorem 1.23. Let (M, g) be a symmetric space for which the non-Euclidean part of its universal cover is split-rank and let $\tau \in \text{Spec}_L(M, g)$. Now let γ be a closed geodesic in (M, g) of length τ with $\gamma'(0) = (v_0, v_1, \dots, v_\ell)$. Then by Proposition 1.18, the Morse index of γ satisfies

$$\sigma_{\Delta M}(\gamma) \equiv h_{(M,g)}(\|v_0\|, \|v_1\|, \dots, \|v_\ell\|, \dim \text{Fix}_\gamma(\Phi_\tau)) \pmod{4},$$

for some function $h_{(M,g)}(x_0, x_1, \dots, x_\ell, y) = f_{(M,g)}(x_0, x_1, \dots, x_\ell) - y + C(M)$.

Now, in the event that the metric g has CLU length spectrum we see that for each $j = 0, 1, \dots, \ell$ $\|v_j\| = c_j(\tau)$. Hence, we may replace the function $h_{(M,g)}$ with $H_{(M,g)} : \text{Spec}_L(M, g) \times \{0, 1, \dots, 2 \dim M - 1\} \rightarrow \{0, 1, 2, 3\}$. In the case where (M, g) satisfies any one of (3)- (5), an examination of the proof of Proposition 1.18 (and recalling Lemma 3.4) reveals that the function $f_{(M,g)}$ is identically zero and we find

$$\sigma_{\Delta M}(\gamma) \equiv H_{(M,g)}(\dim \text{Fix}_\gamma(\Phi_\tau)) \pmod{4}.$$

Finally, suppose $M = G/K$ is a split-rank symmetric space with the metric induced by $-cB \upharpoonright \mathfrak{p} \times \mathfrak{p}$ for some $c > 0$, where B is the Killing form on \mathfrak{g} . Given $v \in \mathfrak{p} \equiv T_o M$, we have

$\|v\|^2 = -cB(v, v) = -c\text{Tr}(\text{ad}(v) \circ \text{ad}(v))$, so that in the case where $v \in \mathfrak{a} \subset \mathfrak{p}$ this becomes:

$$\|v\|^2 = \sum_{\beta \in R^+} \dim \mathfrak{g}^\beta (\pi\beta(v)^2) = -2\pi \sum_{\beta \in R^+} n_\beta \beta(v)^2.$$

Hence, for any $v, w \in \Lambda_Z(M) \subset \mathfrak{a}$ we see $\|v\|^2 = \|w\|^2$ if and only if $\sum_{\beta \in R^+} n_\beta \beta(v)^2 = \sum_{\beta \in R^+} n_\beta \beta(w)^2$, which implies by the lemma that $\sum_{\beta \in R^+} n_\beta \beta(v) \equiv \sum_{\beta \in R^+} n_\beta \beta(w) \pmod{2}$. Now, M is split-rank if and only if $n_\beta = 2\hat{n}_\beta \in 2\mathbb{Z}$ for all $\beta \in R^+$. Then, for any $v, w \in \Lambda_Z(M) \subset \mathfrak{a}$ we see that $\|v\|^2 = \|w\|^2$ if and only if $\sum_{\beta \in R^+} \hat{n}_\beta \beta(v)^2 = \sum_{\beta \in R^+} \hat{n}_\beta \beta(w)^2$, which implies that $\sum_{\beta \in R^+} \hat{n}_\beta \beta(v) \equiv \sum_{\beta \in R^+} \hat{n}_\beta \beta(w) \pmod{2}$. Therefore, we conclude that $\|v\| = \|w\|$ implies that $\sum_{\beta \in R^+} n_\beta \beta(v) \equiv \sum_{\beta \in R^+} n_\beta \beta(w) \pmod{4}$. That is, there is a function $f_{(M,g)}$ such that for any $v \in \bar{\mathcal{C}} \cap \Lambda_I(M) \subset \bar{\mathcal{C}} \cap \Lambda_Z(M)$ we have

$$\sum_{\beta \in R^+} n_\beta \beta(v) \equiv f_{(M,g)}(\tau) \pmod{4},$$

where $\tau = \|v\|$ is the length of the geodesic γ . □

The following examples show that Theorem 1.23 fails for arbitrary symmetric spaces.

Example 3.15 (A semi-simple Lie group for which Theorem 1.23 fails). For any simple Lie group H with Lie algebra \mathfrak{h} , let \bar{g}^H denote the bi-invariant metric induced by $-rB$, where B is the Killing form on \mathfrak{h} and $r > 0$ is chosen to agree with the inner product structure used in Appendix A.2. Now, consider the semi-simple Lie group $U = \text{SU}(2) \times \text{SO}(3)$ equipped with the bi-invariant metric $g = \frac{1}{4}\bar{g}^{\text{SU}(2)} \times \bar{g}^{\text{SO}(3)}$. Since $\text{SU}(2)$ is type A_1 , we can deduce from Equation A.6 that the integral lattice of $U = \text{SU}(2) \times \text{SU}(2)/1 \times Z(\text{SU}(2))$ is given by

$$\Lambda_I(U) = \Lambda_I(\text{SU}(2)) \oplus \Lambda_I(\text{SO}(3)) = \langle (2L_1, 0), (0, L_1) \rangle.$$

The vectors $v = (2L_1, 2L_1), w = (4L_1, L_1) \in \Lambda_I(U)$ belong to the same Weyl chamber \mathcal{C} and are both of length $\tau \equiv \sqrt{5}$ with respect to the metric g . As they are regular vectors in \mathfrak{u} we see that $\dim \text{Fix}_v(\Phi_{\sqrt{5}}) = \dim \text{Fix}_w(\Phi_{\sqrt{5}}) \equiv D$ is of maximal dimension. And, since v and w are the only vectors of length $\sqrt{5}$ in $\Lambda_I(U) \cap \bar{\mathcal{C}}$, Equation 1.6 yields

$$\text{Wave}_0^\bullet(\sqrt{5}) = \left(\frac{1}{2\pi i} \right)^{\frac{D-1}{2}} \left(i^{-\sigma_v} \int_{\text{Fix}_v(\Phi_{\sqrt{5}})} d\mu_v + i^{-\sigma_w} \int_{\text{Fix}_w(\Phi_{\sqrt{5}})} d\mu_w \right),$$

where σ_v (resp. σ_w) is the Morse index of the geodesic γ_v (resp. γ_w) and μ_v (resp. μ_w) is the canonical Duistermaat-Guillemin measure on $\text{Fix}_v(\Phi_\tau)$ (resp. $\text{Fix}_w(\Phi_\tau)$). Now, using Proposition 1.18 and Lemma 3.6, we find that σ_v fulfills

$$\begin{aligned} \sigma_v &\equiv f_{(\text{SU}(2), \frac{1}{4}\bar{g}^{\text{SU}(2)})}(\|2L_1\|) + f_{(\text{SO}(3), \bar{g}^{\text{SO}(3)})}(\|2L_1\|) + D \pmod{4} \\ &\equiv 0 + 0 + D \pmod{4} \\ &\equiv D \pmod{4}. \end{aligned}$$

On the other hand, we see that σ_w satisfies

$$\begin{aligned}\sigma_w &\equiv f_{(\mathrm{SU}(2), \frac{1}{4}\bar{g}^{\mathrm{SU}(2)})}(\|4L_1\|) + f_{(\mathrm{SO}(3), \bar{g}^{\mathrm{SO}(3)})}(\|L_1\|) + D \pmod{4} \\ &\equiv 0 + 2 + D \pmod{4} \\ &\equiv D + 2 \pmod{4}.\end{aligned}$$

Therefore, $\sigma_v - \sigma_w \equiv 2 \pmod{4}$ and we conclude that Theorem 1.23 does not hold. We also see that in the event that $\mathrm{Fix}_v(\Phi_{\sqrt{5}})$ and $\mathrm{Fix}_w(\Phi_{\sqrt{5}})$ have the same volume with respect to their Duistermaat-Guillemin measures, the wave invariant $\mathrm{Wave}_0^\bullet(\sqrt{5})$ is zero. Clearly, similar examples can be constructed for certain reducible homogeneity types $M = \Gamma \backslash (M_0 \times \widetilde{M}_1 \times \cdots \times \widetilde{M}_q) \notin \mathcal{H}$, where \mathcal{H} is as defined in Theorem 1.13.

Example 3.16 (A family of irreducible symmetric spaces for which Theorem 1.23 fails). A symmetric space $M = G/K$ is said to be of *maximal rank* if $\mathrm{rank}(M) = \mathrm{rank}(G)$ or, equivalently, its restricted roots occur with multiplicity one [Lo, Proposition VI.4.1]. Now, for any compact semi-simple Lie group G with maximal torus T there is an involution $\sigma \in \mathrm{Aut}(G)$ such that for any $x \in T$ we have $\sigma(x) = x^{-1}$ [Lo, Theorem V.4.2] and the associated *space of symmetric elements* $G_\sigma \equiv \{x\sigma(x)^{-1} : x \in G\} \subset G$ is a symmetric space of maximal rank with restricted root system isomorphic to that of G ; however, the roots occur with multiplicity *one* instead of two.

Now, let $\sigma \in \mathrm{Aut}(\mathrm{SO}(2n+1))$, $n \geq 2$, be an automorphism of the type discussed in the previous paragraph and let $\mathrm{SO}(2n+1)_\sigma$ denote the corresponding space of symmetric elements. The restricted root system R of $\mathrm{SO}(2n+1)_\sigma$ is of type B_n and the integral lattice is given by $\Lambda_I = \Lambda_Z = \langle e_1, \dots, e_n \rangle$. Consider the vectors $v = (7, 6, 0, \dots, 0)$, $w = (9, 2, 0, \dots, 0) \in \overline{\mathcal{C}} \cap \Lambda_I$ of length $\tau = \sqrt{85}$ and let γ_v and γ_w denote the corresponding closed geodesics of length τ in $\mathrm{SO}(2n+1)_\sigma$. Then, it is clear that $\{\alpha \in R^+ : \alpha(v) = 0\} = \{\alpha \in R^+ : \alpha(w) = 0\} = \{\epsilon_j : 3 \leq j \leq n\} \cup \{\epsilon_\mu \pm \epsilon_\nu : 3 \leq \mu < \nu \leq n\}$, which is a set of order $(n-2)^2$, so that $\deg_{\mathrm{sing}}(v) = \deg_{\mathrm{sing}}(w) = (n-2)^2$ or, equivalently,

$$\dim \mathrm{Fix}_{\gamma_v}(\Phi_\tau) = \dim \mathrm{Fix}_{\gamma_w}(\Phi_\tau) \equiv D,$$

where, as before, for any closed geodesic γ of length τ , $\mathrm{Fix}_\gamma(\Phi_\tau)$ denotes the component of $\mathrm{Fix}(\Phi_\tau)$ containing $\gamma'(t)/\|\gamma'(t)\|$. Using the formula for the sum of the positive roots provided in Section A.2.2 we find that

$$\sum_{\alpha \in R^+} \alpha(v) \equiv \begin{cases} 3 \pmod{4} & \text{for } n \text{ even} \\ 1 \pmod{4} & \text{for } n \text{ odd} \end{cases}$$

and

$$\sum_{\alpha \in R^+} \alpha(w) \equiv \begin{cases} 1 \pmod{4} & \text{for } n \text{ even} \\ 3 \pmod{4} & \text{for } n \text{ odd} \end{cases}$$

It then follows from Equation 2.11 that the Morse indexes of γ_v and γ_w are not congruent modulo 4 (but they do have the same parity). Therefore, it is not the case that the Morse

index modulo 4 of a closed geodesic γ in $\mathrm{SO}(2n+1)_\sigma$ is a function of its length τ and the dimension of the corresponding component of $\mathrm{Fix}(\Phi_\tau)$.

Remark 3.17. It would appear to be of interest to explore whether cancellations can occur in the trace formula for the examples presented above and other symmetric spaces for which Theorem 1.23 fails. This will be taken up in a subsequent article.

4. UNCLEAN LEFT-INVARIANT METRICS ON $\mathrm{SO}(3)$ AND S^3

Theorem 1.8 states that every compact globally symmetric space is clean. In this section we provide a proof of Theorem 1.9 which states that there are homogeneous metrics that fail to be clean. Indeed, within the class of naturally reductive left-invariant metrics on $\mathrm{SO}(3)$, we give an explicit description of the metrics that are clean and unclean. We find that the clean metrics contain a residual set, while the unclean metrics form a dense subset. Therefore, even in the setting of homogeneous Riemannian metrics, a new technique for analyzing the singularities of the wave trace is needed. The unclean metrics we observe satisfy condition (1) of Definition 1.2, but possess periods $\pm\tau$ for which condition (2) is not met. We are able to verify condition (1) by explicitly computing the closed geodesics for these metrics. We are not aware of whether condition (1) is always satisfied for homogeneous metrics. Although we discuss left-invariant naturally reductive metrics on $\mathrm{SO}(3)$, similar statements and arguments apply to the left-invariant naturally reductive metrics on $S^3 \simeq \mathrm{SU}(2)$; i.e., the Berger spheres.

4.1. Classification of Naturally Reductive Metrics on Lie Groups. Let (M, g) be a connected homogeneous Riemannian manifold. Choose a base point $p_0 \in M$. Let G be a transitive group of isometries of (M, g) , and let K be the isotropy group of p_0 . Now, suppose the Lie algebra \mathfrak{g} of G decomposes into a direct sum $\mathfrak{g} = \mathfrak{K} + \mathfrak{p}$, where \mathfrak{K} is the Lie algebra of K and \mathfrak{p} is an $\mathrm{Ad}(K)$ -invariant complement of \mathfrak{K} . Given a vector $X \in \mathfrak{g}$ we obtain a Killing field X^* on M by $X_p^* \equiv \frac{d}{dt}|_{t=0} \exp_H tX \cdot p$ for $p \in M$. The map $X \mapsto X^*$ is an antihomomorphism of Lie algebras. We may identify \mathfrak{p} with $T_{p_0}M$ by the linear map $X \mapsto X_{p_0}^*$. Thus, the homogeneous Riemannian metric g on M corresponds to an inner product $\langle \cdot, \cdot \rangle$ on \mathfrak{p} . For $X \in \mathfrak{g}$, write $X = X_{\mathfrak{K}} + X_{\mathfrak{p}}$ with $X_{\mathfrak{K}} \in \mathfrak{K}$ and $X_{\mathfrak{p}} \in \mathfrak{p}$. Recall that for $X, Y \in \mathfrak{p}$,

$$(4.1) \quad (\nabla_{X^*} Y^*)_{p_0} = -\frac{1}{2}([X, Y]_{\mathfrak{p}}^*)_{p_0} + W(X, Y)_{p_0}^*,$$

where $W : \mathfrak{p} \times \mathfrak{p} \rightarrow \mathfrak{p}$ is the symmetric bilinear map defined by

$$2\langle W(X, Y), Z \rangle = \langle [Z, X]_{\mathfrak{p}}, Y \rangle + \langle X, [Z, Y]_{\mathfrak{p}} \rangle.$$

Definition 4.2. Let (M, g) be a Riemannian homogeneous space and let G be a transitive group of isometries of (M, g) , so that $M = G/K$.

- (1) (M, g) is said to be *reductive* (with respect to G), if there is an $\mathrm{Ad}(K)$ -invariant complement \mathfrak{p} of \mathfrak{K} in \mathfrak{g} .

- (2) (M, g) is said to be *naturally reductive* (with respect to G) or *G -naturally reductive*, if there exists an $\text{Ad}(K)$ -invariant complement \mathfrak{p} of \mathfrak{K} (as above) such that

$$\langle [Z, X]_{\mathfrak{p}}, Y \rangle + \langle X, [Z, Y]_{\mathfrak{p}} \rangle = 0,$$

or equivalently $W \equiv 0$. That is, for any $Z \in \mathfrak{p}$ the map $[Z, \cdot]_{\mathfrak{p}} : \mathfrak{p} \rightarrow \mathfrak{p}$ is skew symmetric with respect to $\langle \cdot, \cdot \rangle$.

- (3) (M, g) is said to be *normal homogeneous*, if there is an $\text{Ad}(G)$ -invariant inner product Q on \mathfrak{g} such that

$$Q(\mathfrak{p}, \mathfrak{K}) = 0 \text{ and } Q \upharpoonright \mathfrak{p} = \langle \cdot, \cdot \rangle.$$

Remark 4.3. If G is a connected group of isometries acting transitively on (M, g) , then (M, g) is reductive with respect to G [KS].

In [DZ], D'Atri and Ziller addressed the problem of classifying the naturally reductive left-invariant metrics on compact Lie groups. Recalling that for any subgroup K of the Lie group U the natural action of $G \equiv U \times K$ on U is defined by $(g, k) \cdot x = g x k^{-1}$, D'Atri and Ziller's classification of such metrics is as follows.

Theorem 4.4 ([DZ] Theorems 3 and 7). *Consider a connected compact simple Lie group U and let g_0 be the bi-invariant Riemannian metric on U induced by the negative of the Killing form B . Let $K \leq U$ be a connected subgroup with Lie algebra $\mathfrak{K} = \mathfrak{K}_0 \oplus \mathfrak{K}_1 \oplus \cdots \oplus \mathfrak{K}_r$, where $\mathfrak{K}_0 = Z(\mathfrak{K})$ is the center of \mathfrak{K} and $\mathfrak{K}_1, \dots, \mathfrak{K}_r$ are the simple ideals in \mathfrak{K} . Let \mathfrak{m} be a g_0 -orthogonal complement of \mathfrak{K} in \mathfrak{u} . Given any $\alpha, \alpha_1, \dots, \alpha_r > 0$ and an arbitrary inner product h on \mathfrak{K}_0 , let $g_{\alpha, \alpha_1, \dots, \alpha_r, h}$ denote the metric on U induced by the $\text{Ad}(K)$ -invariant inner product on \mathfrak{u} given by*

$$(4.5) \quad \alpha g_0 \upharpoonright \mathfrak{m} \oplus h \upharpoonright \mathfrak{K}_0 \oplus \alpha_1 g_0 \upharpoonright \mathfrak{K}_1 \oplus \cdots \oplus \alpha_r g_0 \upharpoonright \mathfrak{K}_r.$$

Then:

- (1) $g_{\alpha, \alpha_1, \dots, \alpha_r, h}$ is naturally reductive with respect to the natural action of $G \equiv U \times K$ on U ;
- (2) every left-invariant naturally reductive metric on U arises in this fashion;
- (3) $g_{\alpha, \alpha_1, \dots, \alpha_r, h}$ is normal homogeneous if and only if $h \leq \alpha g_0 \upharpoonright \mathfrak{K}$.
- (4) $\text{Isom}(g_{\alpha, \alpha_1, \dots, \alpha_r, h})^0$, the connected isometry group, is given by $U \times N_U(K)^0$, where $N_U(K)$ denotes the normalizer of K in U .

Naturally reductive spaces generalize the notion of a symmetric space and, as is the case with symmetric spaces, the geodesics in a naturally reductive space $(M = G/K, g)$ are of the form $\exp_G(tX) \cdot p$, where $X \in \mathfrak{p}$. That is, the geodesics in a naturally geodesic space are precisely the integral curves of Killing vector fields.

Proposition 4.6. *Let U be a simple Lie group and K a connected subgroup. Now, let $g_{\alpha, \alpha_1, \dots, \alpha_r, h}$ be a $U \times K$ naturally reductive metric on U and $\mathfrak{p} \leq \mathfrak{u} \times \mathfrak{k}$ the $\text{Ad}(\Delta K)$ -invariant complement*

constructed in [Appendix B](#). Then, the geodesics through $g \in U$ with respect to $g_{\alpha, \alpha_1, \dots, \alpha_r, h}$ are of the form

$$\exp_{U \times K}(t \operatorname{Ad}(g)X, Y) \cdot g = g \exp_U(tX) \exp_U(-tY),$$

where $(X, Y) \in \mathfrak{p}$, and such a geodesic is smoothly closed if and only if $\exp_U(tX) = \exp_U(tY)$ for some $t > 0$.

This will prove useful in our proof of [Theorem 1.9](#), where we will need an explicit description of the closed geodesics with respect to a left-invariant naturally reductive metric on $\operatorname{SO}(3)$.

Since, the geodesics in a naturally reductive space are integral curves of Killing fields, there are no geodesic lassos in a naturally reductive space (i.e., all self-intersections of a geodesic are smooth). Although it is not needed elsewhere in the paper, we note that since every homogeneous Riemannian space can be expressed as a reductive space [[KS](#), Proposition 1], an application of Noether's theorem (cf. [[Tak](#), Theorem 1.3]) shows there are no geodesic lassos in an arbitrary homogeneous space.

Proposition 4.7. *Let (M, g) be a homogeneous Riemannian manifold and $\gamma : \mathbb{R} \rightarrow M$ a geodesic. If $\gamma(t_0) = \gamma(t_1)$, then $\gamma'(t_0) = \gamma'(t_1)$. That is, any self-intersection of a geodesic in a homogeneous space is smooth.*

4.2. The Poincaré map of naturally reductive metrics. We recall that given a Riemannian manifold (M, g) the geodesic flow is the map $\Phi : \mathbb{R} \times TM \rightarrow TM$ given by

$$\Phi(t, v) = \frac{d}{dt} \gamma_v(t),$$

where γ_v is the unique geodesic with $\gamma'_v(0) = v$. Throughout we will set $\Phi_t(v) = \Phi(t, v)$. Of particular interest to us is the derivative of Φ_τ . If for each $v \in TM$ we let $T_v TM = \mathcal{H}_v \oplus \mathcal{V}_v$ be the decomposition into the horizontal and vertical spaces, then for any $(A, B) \in T_v TM$ we have

$$\Phi_{t*}(A, B) = (Y(t), \nabla Y(t)),$$

where $Y(t)$ is the Jacobi field along γ_v such that $Y(0) = A$ and $\nabla Y(0) = B$ (see [[Sa](#), p. 56]). If the geodesic γ_v is periodic of period τ , then we set

$$P = \Phi_{\tau*} : T_v TM \rightarrow T_v TM.$$

Since $\gamma'_v(t)$ and $t\gamma'_v(t)$ are Jacobi fields along γ_v we see that

$$P(v, 0) = (v, 0) \text{ and } P(0, v) = (\tau v, v).$$

Hence, in order to understand P we must analyze how it behaves on the orthogonal complement of $(v, 0)$ and $(0, v)$; that is, we seek to understand

$$P : E \oplus E \rightarrow E \oplus E,$$

where $E = \{u \in T_p M : \langle u, v \rangle = 0\}$. This map is called the (linearized) *Poincaré map* and from the above if Y is a Jacobi field with initial data $(Y(0), \nabla Y(0)) \in E \oplus E$, then

$$P(Y(0), \nabla Y(0)) = (Y(\tau), \nabla Y(\tau)).$$

In the case of (compact) naturally reductive manifolds the Poincaré map has been completely determined by Ziller as follows.

Let $M = G/K$ be a naturally reductive space and as before let $\mathfrak{p} \leq \mathfrak{g}$ be an $\text{Ad}(K)$ -invariant complement. For any unit vector $v \in \mathfrak{p} \equiv T_{p_0} M$ we let $\gamma_v(t)$ be the unit speed geodesic given by $\exp_G(tv) \cdot p_0$. Now, let $v \in \mathfrak{p}$ be a unit vector such that the geodesic $\gamma_v(t)$ is closed and set $E = \{u \in \mathfrak{p} : \langle u, v \rangle = 0\}$. Then the restriction of the maps $B(\cdot) = -[v, [v, \cdot]_{\mathfrak{K}}]$ and $T(\cdot) = -[v, \cdot]_{\mathfrak{p}}$ to E are symmetric and skew-symmetric, respectively. Now let E_0 denote the 0-eigenspace of $B : E \rightarrow E$ and E_1 be the sum of its non-zero eigenspaces, and we express E_0 as the orthogonal direct sum $E_0 = E_2 \oplus E_3$, where $E_2 = \{X \in E_0 : T(X) \in E_1\}$. Then as in [Z2, p. 579] we define the following subspaces of $E \oplus E$:

- (1) $V_1 = \{(X, \frac{1}{2}[X, v]_{\mathfrak{p}}) : X \in E_1 \oplus E_3\}$
- (2) $V_2 = \{(0, X) : X \in E_1\}$
- (3) $V_3 = \{(X, \frac{1}{2}[v, X]_{\mathfrak{p}}) : X \in E_2\}$
- (4) $V_4 = \{(X, \frac{1}{2}[v, X]_{\mathfrak{p}}) : X \in E_3\} = \{(X, -\frac{1}{2}T(X)) : X \in E_3\}$
- (5) $V_5 = \{(Z, X + \frac{1}{2}[v, Z]_{\mathfrak{p}}) : X \in E_2, Z \in E_1 \text{ and } B(Z) = T(X) \equiv [X, v]_{\mathfrak{p}}\}$

Remark 4.8.

- (1) In [Z2] there is an omission in the definition of V_5 (cf. [Z1, p. 73]).
- (2) We note that since $B : E_1 \rightarrow E_1$ is an isomorphism, V_5 is non-trivial if and only if E_2 is non-trivial. In particular, for each $X \in E_2$, there exists a unique $Z \in E_1$ such that $B(Z) = T(X)$.
- (3) It will be useful later to notice that $E_1 \leq [\mathfrak{K}, v]$. Indeed, following [Z1, p. 72], we recall that $B : E \rightarrow E$ is a self-adjoint map. Let X_1, \dots, X_q be an orthonormal basis of eigenvectors with eigenvalues $\lambda_1, \dots, \lambda_q$, and set $Z_i \equiv [v, X_i]_{\mathfrak{K}} \in \mathfrak{K}$. Then $\lambda_i X_i = B(X_i) = [Z_i, v]$ and for $\lambda_i \neq 0$ we get $X_i = \frac{1}{\lambda_i}[Z_i, v] \in [\mathfrak{K}, v]$, which establishes the claim.

With the notation as above we have the following theorem due to Ziller.

Theorem 4.9. *Let $(M = G/K, g)$ be a (compact) naturally reductive space and let $\gamma_v(t) = \exp_G(tv) \cdot p_0$ be a smoothly closed unit speed geodesic in M of length τ with $\gamma'_v(0) = v \in \mathfrak{p} \equiv T_{p_0} M$. Then*

- (1) ([Z2, Theorem 1]) $E \oplus E = V_1 \oplus V_2 \oplus V_3 \oplus V_4 \oplus V_5$
- (2) ([Z1, Theorem 1]) *The Poincaré map $P : E \oplus E \rightarrow E \oplus E$ along γ_v is described as follows:*
 - (a) $P \upharpoonright V_1 \oplus V_2 \oplus V_3 = \text{Id};$

- (b) $P(X, \frac{1}{2}[v, X]_{\mathfrak{p}}) = (\Psi(X), \Psi(\frac{1}{2}[v, X]_{\mathfrak{p}})) = (\Psi(X), \frac{1}{2}[v, \Psi(X)]_{\mathfrak{p}})$, for $(X, \frac{1}{2}[v, X]_{\mathfrak{p}}) \in V_4$, where Ψ is the isometry $e^{\text{ad}(\tau v)} = \text{Ad}(\exp_H(\tau v))$, we recall that because γ_v is a geodesic it is given by $\exp_H(tv) \cdot p_0$ and since it is closed of length τ we have that $\exp_H(\tau v) \in K$;
- (c) $P(Z, X + \frac{1}{2}[v, Z]_{\mathfrak{p}}) = \tau(X, \frac{1}{2}[v, X]_{\mathfrak{p}}) + (Z, \frac{1}{2}[v, Z]_{\mathfrak{p}})$, for $(Z, X + \frac{1}{2}[v, Z]_{\mathfrak{p}}) \in V_5$.

Remark 4.10. The compactness condition in the above was used by Ziller to establish that a Jacobi field $J(t)$ along γ_v with $J(0) \in V_5$ must have unbounded length, which is used to show that $V_5 \cap (V_1 \oplus V_2 \oplus V_3 \oplus V_4)$ is trivial [Z2, p. 579-80]. However, this argument only really requires completeness, which is enjoyed by all naturally reductive spaces since geodesics are precisely the orbits of one-parameter groups of isometries. Therefore, the above is true for all naturally reductive manifolds.

The following observation is an immediate consequence of the previous proposition.

Corollary 4.11. *Let $\gamma_v(t)$ be a closed unit speed geodesic as above and let $Y(t)$ be a Jacobi field along γ_v . Then $Y(t)$ is periodic if and only if $Y(t)$ has the following initial conditions:*

$$(Y(0), \nabla Y(0)) \in V_1 \oplus V_2 \oplus V_3 \oplus V_4^{\text{per}} \oplus \text{Span}_{\mathbb{R}}\{(v, 0)\},$$

where $V_4^{\text{per}} \equiv \{(X, \frac{1}{2}[v, X]_{\mathfrak{p}}) : X \in E_3 \text{ and } \psi(X) = X\} \leq V_4$.

4.3. Proof of Theorem 1.9. Let U be an arbitrary compact semi-simple Lie group with bi-invariant metric g_0 induced by the negative of the Killing form B on $T_e U$. Now for any left-invariant metric g on U there is a linear transformation $\Omega : T_e U \rightarrow T_e U$ that is self-adjoint with respect to $-B$ and such that for any $v, w \in T_e U$ we have $\langle v, w \rangle = -B(\Omega(v), w)$, where $\langle \cdot, \cdot \rangle$ is the restriction of g to $T_e U$.

Definition 4.12. With the notation as above, the eigenvalues $0 < \mu_1 \leq \mu_2 \leq \dots \leq \mu_n$ of Ω are called the *eigenvalues of the metric g* .

Proposition 4.13 ([BFSTW] Proposition 3.2). *Two left-invariant metrics g_1 and g_2 on $\text{SO}(3)$ are isometric if and only if g_1 and g_2 have the same eigenvalues counting multiplicities.*

Notation and Remarks 4.14. We will now establish notation and collect some facts that will prove useful throughout the remainder of this section.

- (1) For the remainder of this section we will let U denote the Lie group $\text{SO}(3)$, \mathfrak{u} denote its Lie algebra $\mathfrak{so}(3)$, and g_0 will denote the bi-invariant metric on $\text{SO}(3)$ induced by $-B$, where B denotes the Killing form. Additionally, we will let \exp denote the exponential map $\exp_U : \mathfrak{u} \rightarrow U$.
- (2) With Proposition 4.13 in mind we let

$$\Theta_1 = \frac{1}{2\sqrt{2}} \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}, \quad \Theta_2 = \frac{1}{2\sqrt{2}} \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, \quad \Theta_3 = \frac{1}{2\sqrt{2}} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

denote the standard g_0 -orthonormal basis of $\mathfrak{so}(3) \simeq \mathfrak{su}(2)$. Then for any choice of positive constants c_1, c_2 and c_3 the self-adjoint map $\Omega : (\mathfrak{so}(3), -B) \rightarrow (\mathfrak{so}(3), -B)$ given by $\Omega(\Theta_j) = c_j \Theta_j$ defines a left-invariant metric $g_{(c_1, c_2, c_3)}$ on $\mathrm{SO}(3)$ and, by Proposition 4.13, these account for all of the left-invariant metrics on $\mathrm{SO}(3)$ up to isometry. Now, since $\mathrm{SO}(2)$ is the only non-trivial connected proper subgroup of $\mathrm{SO}(3)$ it follows from Theorem 4.4 that up to isometry the left-invariant naturally reductive metrics on $\mathrm{SO}(3)$ are the metrics $g_{(\alpha, \alpha, A)}$ given by:

$$(4.15) \quad g_{(\alpha, \alpha, A)} = \alpha g_0 \upharpoonright \mathfrak{m} \oplus A g_0 \upharpoonright \mathfrak{K},$$

where $\mathfrak{K} = \mathfrak{so}(2) = \mathrm{Span}(\Theta_3)$ and $\mathfrak{m} = \mathfrak{K}^{\perp_0} = \mathrm{Span}\{\Theta_1, \Theta_2\}$ is the orthogonal complement of \mathfrak{K} with respect to g_0 . We set $K = \exp_U(\mathfrak{K})$.

- (3) Let \mathfrak{p} denote the $\mathrm{Ad}(K)$ -invariant complement of $\Delta\mathfrak{K} \leq \mathfrak{g} \times \mathfrak{K}$ discussed in Appendix B. Then we have the following.

- (a) If $\alpha = A$, then by Equation B.2 we see $\mathfrak{p} = \mathfrak{g} \oplus 0$. In which case

$$\mathfrak{p} = \mathrm{Span}\left\{\frac{1}{\sqrt{\alpha}}(\Theta_1, 0), \frac{1}{\sqrt{\alpha}}(\Theta_2, 0), \frac{1}{\sqrt{\alpha}}(\Theta_3, 0)\right\}$$

and

$$\Delta\mathfrak{K} = \mathrm{Span}\{D = (\Theta_3, \Theta_3)\},$$

where by $\mathrm{Span}\{A_1, \dots, A_k\}$ we denote the linear span of A_1, \dots, A_k over \mathbb{R} .

- (b) If $\alpha \neq A$, then by Equation B.1 $\mathfrak{p} = \mathfrak{p}_1 \oplus \mathfrak{q}_0$, where $\mathfrak{p}_1 = \{(X, 0) : X \in \mathfrak{u} = \mathfrak{K}^{\perp_0}\}$ and $\mathfrak{q}_0 = \{(\bar{A}Z, -\alpha Z) : Z \in \mathfrak{K}\}$ for $\bar{A} = \frac{A\alpha}{\alpha - A}$. In which case

$$\mathfrak{p} = \mathrm{Span}\left\{Z_1 = \left(\frac{1}{\sqrt{\alpha}}\Theta_1, 0\right), Z_2 = \left(\frac{1}{\sqrt{\alpha}}\Theta_2, 0\right), Z_3 = \frac{1}{\sqrt{A(\bar{A} + \alpha)}}(\bar{A}\Theta_3, -\alpha\Theta_3)\right\}$$

and

$$\Delta\mathfrak{K} = \mathrm{Span}\{D = (\Theta_3, \Theta_3)\}.$$

It is clear that the adjoint action of $\Delta K \leq G \times K$ on \mathfrak{p} fixes Z_3 and acts as the group of rotations on $\mathrm{Span}_{\mathbb{R}}\{Z_1, Z_2\} = \mathfrak{p}_1$.

- (4) For any $(V, W) \in \mathfrak{p}$, where \mathfrak{p} is as above, the geodesic $\gamma_{(V, W)}(t)$ with $\gamma_{(V, W)}(0) = e$ and $\gamma'_{(V, W)}(0) = V - W$ is given by

$$\gamma_{(V, W)}(t) = \exp(tV) \exp(-tW).$$

The geodesic $\gamma_{(V, W)}$ is a one-parameter subgroup of $\mathrm{SO}(3)$ if and only if $V, W \in \mathfrak{so}(3)$ are linearly dependent.

- (5) For any compact Lie group endowed with a bi-invariant metric the sectional curvature of a 2-plane σ in the Lie algebra spanned by two orthonormal vectors X and Y is given by $\mathrm{Sec}(\sigma) = \frac{1}{4} \| [X, Y] \|^2$. Consequently, with respect to the metric g_0 , the Lie group $\mathrm{SO}(3)$ has constant sectional curvature $\frac{1}{8}$ and is double covered by $S^3(2\sqrt{2})$, the round 3-sphere of radius $2\sqrt{2}$. It follows that the geodesics in $(\mathrm{SO}(3), g_0)$ are all closed, have a common (primitive) length $\ell_0 \equiv 2\sqrt{2}\pi$.

- (6) It follows from the previous remark that any two primitive geodesics through a given point of $\mathrm{SO}(3)$ with respect to $g_0 = g_{(1,1,1)}$ have only one point in common or have exactly the same image. Furthermore, since g_0 is bi-invariant, its geodesics through e coincide with the one-parameter subgroups of $\mathrm{SO}(3)$. Given a vector $X \in \mathfrak{u} = \mathfrak{so}(3)$ we then define its *period* to be $\mathrm{Per}(X) = \frac{\ell_0}{\|X\|_0}$, so $\mathrm{Per}(X)$ is the amount of time it takes for the one-parameter subgroup $\exp(tX)$ to return to the identity element for the first time.
- (7) It will be useful to observe that $\mathrm{vol}(g_{(\alpha,\alpha,A)}) = \alpha\sqrt{A}V_0$, where $V_0 \equiv \mathrm{vol}(g_{(1,1,1)}) = \frac{1}{2} \mathrm{vol}(S^3(2\sqrt{2})) = 16\sqrt{2}\pi^2$.

We now describe the closed geodesics of an arbitrary naturally reductive metric on $\mathrm{SO}(3)$ and compute the length spectrum.

Theorem 4.16. *Consider the naturally reductive metric $g_{(\alpha,\alpha,A)}$ on $\mathrm{SO}(3)$ and let ℓ_0 be as in 4.14(5).*

- (1) *If $\alpha = A$, then the closed geodesics through the identity are precisely the one-parameter subgroups of $\mathrm{SO}(3)$ and the non-trivial primitive geodesics are all of length $\sqrt{A}\ell_0$.*
- (2) *If $A \neq \alpha$, then the geodesic $\gamma_{(V,W)}$ is closed if and only if one of the following holds:*
 - (a) *$(V,W) \in \mathfrak{p}_1$, in which case $\gamma_{(V,W)}$ is a one-parameter subgroup of $\mathrm{SO}(3)$ with primitive length $\sqrt{\alpha}\ell_0$.*
 - (b) *$(V,W) \in \mathfrak{q}_0$, in which case $\gamma_{(V,W)}$ is a one-parameter subgroup of $\mathrm{SO}(3)$ with primitive length $\sqrt{A}\ell_0$.*
 - (c) *$(V,W) = (X + \overline{A}Z, -\alpha Z) \in \mathfrak{p}$, where $X \neq 0 \in \mathfrak{u}$ and $Z \neq 0 \in \mathfrak{K}$ and there exist $p, q \in \mathbb{N}$ relatively prime integers such that:*
 - (i) $\frac{q^2}{p^2} > \frac{A^2}{(A-\alpha)^2}$
 - (ii) $\|X\|_0^2 = \sigma(p, q, \alpha, A)\|Z\|_0^2$, where $\sigma(p, q, \alpha, A) \equiv \frac{q^2\alpha^2}{p^2} - \frac{A^2\alpha^2}{(\alpha-A)^2}$.*In this case we see that the closed geodesic $\gamma_{(V,W)}$ is not a one-parameter subgroup and its primitive length is given by $\sqrt{\alpha}\ell_0[q^2 + p^2\frac{A}{\alpha-A}]^{\frac{1}{2}}$, which is always strictly larger than $\sqrt{\alpha}\ell_0$.*

Consequently, the length spectrum of $g_{(\alpha,\alpha,A)}$ is given by

$$\mathrm{Spec}_L(g_{(\alpha,\alpha,A)}) = \begin{cases} \{0\} \cup \{k\sqrt{\alpha}\ell_0 : k \in \mathbb{N}\} & \alpha = A \\ \{0\} \cup \{k\sqrt{\alpha}\ell_0, k\sqrt{A}\ell_0, k\tau : k \in \mathbb{N} \text{ and } \tau > 0 \text{ with } \mathcal{E}_{\tau,\alpha,A} \neq \emptyset\} & A \neq \alpha, \end{cases}$$

where for each $\tau > 0$ we let $\mathcal{E}_{\tau,\alpha,A}$ denote the finite collection of relatively prime ordered pairs $(p, q) \in \mathbb{N} \times \mathbb{N}$ satisfying $\frac{q}{p} > |\frac{A}{A-\alpha}|$ and $\sqrt{\alpha}\ell_0[q^2 + p^2\frac{A}{\alpha-A}]^{\frac{1}{2}} = \tau$.

Definition 4.17. Let $g_{(\alpha,\alpha,A)}$ be a naturally reductive metric on $\mathrm{SO}(3)$ with $\alpha \neq A$.

- (1) A geodesic of the form given in Theorem 4.16(2a) or a translate thereof is said to be of *Type I*.

- (2) A geodesic of the form given in Theorem 4.16(2b) or a translate thereof is said to be of *Type II*.
- (3) A geodesic of the form given in Theorem 4.16(2c) or a translate thereof is said to be of *Type III*.

A periodic orbit of the geodesic flow of $g_{(\alpha, \alpha, A)}$ will be called Type I, II or III according to whether its corresponding closed geodesic is.

Remark 4.18. The systole of a closed Riemannian manifold (M, g) , denoted $\text{Syst}(M, g)$, is the minimum length of a non-contractible closed geodesic and is necessarily at least as large as $\tau_{\min}(M, g)$, the minimum length of a non-trivial closed geodesic. Theorem 4.16 shows us that if $\alpha \neq A$, then the shortest non-trivial closed geodesic with respect to $g_{(\alpha, \alpha, A)}$ is always of Type I or Type II. Therefore, since (primitive) one-parameter subgroups of $\text{SO}(3)$ are homotopically non-trivial, it follows that for any $\alpha, A > 0$, we have $\text{Syst}(g_{(\alpha, \alpha, A)}) = \tau_{\min}(g_{(\alpha, \alpha, A)})$. We also note that it is easy to show that a prime geodesic of Type III is homotopically trivial if and only if $p + q$ is even.

Remark 4.19. In the case where $A \leq \alpha$ the primitive geodesics of Type I and II are shorter than the primitive geodesics of Type III. However, when $A > \alpha$, this need not be the case. For example, if we let $\alpha = 1$ and $A = 10$, then $(p, q) = (1, 2)$ gives rise to a primitive geodesic that is not a one-parameter subgroup and is of length $\ell_0 \sqrt{4 + \frac{10}{9}}$. However, if $A > \alpha$ and $(A - \alpha)^2 < \alpha$, then the prime geodesics of Type I and II will still be shorter than the prime geodesics of Type III.

Proof of Theorem 4.16. For any vector $U \in TG$ we will let $\|U\|_0$ (respectively $\|U\|$) denote its length with respect to the metric g_0 (respectively $g_{(\alpha, \alpha, A)}$).

In the case where $\alpha = A$ we recall from 4.14 that $\mathfrak{p} = \mathfrak{p}'_1 = \mathfrak{g} \oplus 0$. Hence, the geodesics $\gamma_{(V, 0)}(t) = \exp(tV)$ are one-parameter subgroups of G and the primitive non-trivial geodesics are of length $\sqrt{A}\ell_0 = \sqrt{\alpha}\ell_0$ with respect to $g_{(A, A, A)}$. Thus establishing (1).

In the case where $\alpha \neq A$ we recall that $\mathfrak{p} = \mathfrak{p}_1 \oplus \mathfrak{q}_0$, where $\mathfrak{p}_1 = \{(X, 0) : X \in \mathfrak{K}^{\perp_0}\}$ and $\mathfrak{q}_0 = \{(\bar{A}Z, -\alpha Z) : Z \in \mathfrak{K}\}$ (since \mathfrak{K} is abelian). To find the closed geodesics and their lengths we consider the following three cases.

Case I: $(V, W) = (X, 0) \in \mathfrak{p}_1$ for some $X \neq 0 \in \mathfrak{K}^{\perp_0}$.

In this case the geodesic $\gamma_{(V, W)}(t) = \exp(tX)$ is a non-trivial one-parameter subgroup of $\text{SO}(3)$. Consequently, it is closed and has primitive length $L(\gamma_{(V, W)}) = \text{Per}(X) \cdot \|X\| = \text{Per}(X)\sqrt{\alpha}\|X\|_0 = \sqrt{\alpha}\ell_0$.

Case II: $(V, W) = (\bar{A}Z, -\alpha Z) \in \mathfrak{q}_0$ for some $Z \neq 0 \in \mathfrak{K}$

In this case the geodesic $\gamma_{(V, W)}(t) = \exp_G(t(\bar{A} + \alpha)Z)$ is a non-trivial one-parameter subgroup of $\text{SO}(3)$. Consequently, it is closed and has primitive length $L(\gamma_{(V, W)}) = \text{Per}((\bar{A} + \alpha)Z) \cdot \|(\bar{A} + \alpha)Z\| = \text{Per}((\bar{A} + \alpha)Z)\sqrt{A}\|(\bar{A} + \alpha)Z\|_0 = \sqrt{A}\ell_0$.

Case III: $(V, W) = (X + \overline{A}Z, -\alpha Z)$, where $X \neq 0 \in \mathfrak{K}^{\perp_0}$ and $Z \neq 0 \in \mathfrak{K}$.

The geodesic $\gamma_{(V,W)}(t) = \exp(t(X + \overline{A}Z)) \exp(t\alpha Z)$ is clearly not a one-parameter subgroup of $\text{SO}(3)$, and it is closed if and only if there is a $t_0 > 0$ such that

$$(4.20) \quad \exp(t_0(X + \overline{A}Z)) = \exp(-t_0\alpha Z).$$

As noted in 4.14(5), the images of two non-trivial one-parameter subgroups $\exp(tX_1)$ and $\exp(tX_2)$ in $\text{SO}(3)$ either have only the identity element in common or are identical, and the latter occurs if and only if X_1 and X_2 are linearly dependent. Therefore, since $X + \overline{A}Z$ and αZ are linearly independent we see that Equation 4.20 holds if and only if there is a $t_0 > 0$ such that

$$(4.21) \quad e^{t_0(X + \overline{A}Z)} = e^{-t_0\alpha Z} = e,$$

which is equivalent to the existence of relatively prime integers $p, q \in \mathbb{N}$ such that $p \text{Per}(\alpha Z) = q \text{Per}(X + \overline{A}Z)$. Writing out the period of αZ and $X + \overline{A}Z$ explicitly we find that Equation 4.21 holds if and only if there exist relatively prime $p, q \in \mathbb{N}$ such that

- (1) $\sigma(p, q, \alpha, A) \equiv \alpha^2 \left(\frac{q^2}{p^2} - \frac{A^2}{(\alpha - A)^2} \right) > 0;$
- (2) $\|X\|_0^2 = \sigma(p, q, \alpha, A) \|Z\|_0^2.$

The function σ has the property that $\sigma(p, q, \alpha, A) = \sigma(\tilde{p}, \tilde{q}, \alpha, A)$ if and only if $\frac{q}{p} = \frac{\tilde{q}}{\tilde{p}}$ and clearly $\sigma(p, q, \alpha, A) > 0$ is equivalent to $\frac{q^2}{p^2} > \frac{A^2}{(A - \alpha)^2}$.

Now, let $X \neq 0 \in \mathfrak{u}$, $Z \neq 0 \in \mathfrak{K}$ and let $p, q \in \mathbb{N}$ be relatively prime integers such that Equation 4.21 holds. Then $\gamma_{(V,W)}$ is closed and its primitive length is given by

$$\begin{aligned} L(\gamma_{(X + \overline{A}Z, -\alpha Z)})^2 &= [q \text{Per}(X + \overline{A}Z) \|X + (\overline{A} + \alpha)Z\|]^2 \\ &= \left[\frac{q\ell_0}{\|X + \overline{A}Z\|_0} \|X + (\overline{A} + \alpha)Z\| \right]^2 \\ &= \frac{q^2\ell_0^2}{\|X\|_0^2 + \overline{A}^2\|Z\|_0^2} \|X + (\overline{A} + \alpha)Z\|^2 \\ &= \frac{q^2\ell_0^2}{\|X\|_0^2 + \overline{A}^2\|Z\|_0^2} (\|X\|^2 + (\overline{A} + \alpha)^2\|Z\|^2) \\ &= \frac{q^2\ell_0^2}{\|X\|_0^2 + \overline{A}^2\|Z\|_0^2} (\alpha\|X\|_0^2 + (\overline{A} + \alpha)^2 A \|Z\|_0^2) \\ &= \frac{q^2\ell_0^2}{\|X\|_0^2 + \overline{A}^2\|Z\|_0^2} \left(\alpha\|X\|_0^2 + \left(\frac{\alpha^2}{\alpha - A} \right)^2 A \|Z\|_0^2 \right) \\ &= q^2\ell_0^2 \frac{(\alpha\|X\|_0^2 + (\frac{\alpha^2}{\alpha - A})^2 A \|Z\|_0^2)}{\|X\|_0^2 + \overline{A}^2\|Z\|_0^2} \end{aligned}$$

$$\begin{aligned}
&= q^2 \ell_0^2 \frac{(\alpha \|X\|_0^2 + \frac{A\alpha^4}{(\alpha-A)^2} \|Z\|_0^2)}{\|X\|_0^2 + \frac{A^2}{(\alpha-A)^2} \|Z\|_0^2} \\
&= \alpha q^2 \ell_0^2 \cdot \frac{\|X\|_0^2 + \frac{A\alpha^3}{(\alpha-A)^2} \|Z\|_0^2}{\|X\|_0^2 + \frac{A^2\alpha^2}{(\alpha-A)^2} \|Z\|_0^2} \\
&= \alpha q^2 \ell_0^2 \cdot \frac{\|X\|_0^2 + \frac{A\alpha^3}{(\alpha-A)^2 \sigma(p,q,\alpha,A)} \|X\|_0^2}{\|X\|_0^2 + \frac{A^2\alpha^2}{(\alpha-A)^2 \sigma(p,q,\alpha,A)} \|X\|_0^2} \\
&= \alpha q^2 \ell_0^2 \cdot \frac{\|X\|_0^2 + \frac{A\alpha^3}{(\alpha-A)^2} \|Z\|_0^2}{\|X\|_0^2 + \frac{A^2\alpha^2}{(\alpha-A)^2} \|Z\|_0^2} \\
&= \alpha q^2 \ell_0^2 \cdot \frac{\|X\|_0^2 + \frac{A\alpha^3}{(\alpha-A)^2 \sigma(p,q,\alpha,A)} \|X\|_0^2}{\|X\|_0^2 + \frac{A^2\alpha^2}{(\alpha-A)^2 \sigma(p,q,\alpha,A)} \|X\|_0^2} \\
&= \alpha q^2 \ell_0^2 \cdot \frac{1 + \frac{A\alpha p^2}{q^2(\alpha-A)^2 - p^2 A^2}}{1 + \frac{A^2 p^2}{q^2(\alpha-A)^2 - p^2 A^2}} \\
&= \alpha q^2 \ell_0^2 \cdot (1 + \frac{p^2 A(\alpha - A)}{q^2(\alpha - A)^2}) \\
&= \alpha \ell_0^2 \cdot (q^2 + p^2 \frac{A}{(\alpha - A)}).
\end{aligned}$$

In the event that $\alpha > A$ it is clear that this geodesic will have length strictly greater than $\sqrt{\alpha} \ell_0$. To handle the case where $\alpha < A$ we note that $\alpha \ell_0^2 \cdot (q^2 + p^2 \frac{A}{(\alpha-A)})$ is greater than $\alpha \ell_0^2$ if and only if $\frac{A}{A-\alpha} = |\frac{A}{A-\alpha}| < \frac{q^2-1}{p^2}$. But we recall that $p, q \in \mathbb{N}$ were chosen so that $\frac{q}{p} > |\frac{A}{A-\alpha}| = \frac{A}{A-\alpha} > 1$, and notice that for $q > p$ we have $\frac{q^2-1}{p^2} > \frac{q}{p}$. Hence, for $A \neq \alpha$, we see that $L(\gamma_{(V,W)}) > \sqrt{\alpha} \ell_0$.

Cases I-III establish statement (2) of the theorem and the statement concerning the length spectrum of an arbitrary naturally reductive metric $g_{(\alpha,\alpha,A)}$ is now immediate. We conclude the proof by showing that the set $\mathcal{E}_{\tau,\alpha,A}$ is finite.

Indeed, in the case where $A < \alpha$, we see that $\mathcal{E}_{\tau,\alpha,A}$ is a subset of the intersection of an ellipse with the integer lattice in \mathbb{R}^2 , which implies it is finite. In the event that $A > \alpha$, the points $(p, q) \in \mathcal{E}_{\tau,\alpha,A}$ are a subset of the intersection of the integral lattice with the hyperbola

$$\frac{y^2}{\tau^2/\alpha \ell_0^2} - \frac{x^2}{\tau^2(A-\alpha)/\alpha \ell_0^2 A} = 1$$

having asymptotes $y = \pm \sqrt{\frac{A}{A-\alpha}} x$. Now, suppose $\mathcal{E}_{\tau,\alpha,A}$ is infinite, then, since $\frac{q}{p} > |\frac{A}{A-\alpha}| = \frac{A}{A-\alpha} > 1$, we see that q must become arbitrarily large. Then, since the hyperbola is asymptotic to $y = \sqrt{\frac{A}{A-\alpha}} x$, we see that the expression $|p - \sqrt{\frac{A-\alpha}{A}} q|$ can be made arbitrarily small in $\mathcal{E}_{\tau,\alpha,A}$.

However, $\frac{q}{p} > \frac{A}{A-\alpha} > 1$ implies

$$p < \frac{A-\alpha}{A}q < \sqrt{\frac{A-\alpha}{A}}q$$

for any $(p, q) \in \mathcal{E}_{\tau, \alpha, A}$, which implies the quantity $|p - \sqrt{\frac{A-\alpha}{A}}q|$ cannot be made arbitrarily small. So, we see $\mathcal{E}_{\tau, \alpha, A}$ is finite. \square

For any period τ of the geodesic flow of a symmetric metric $g_{(\alpha, \alpha, \alpha)}$ on $\text{SO}(3)$, we see that $\text{Fix}(\Phi_\tau)$ is the entire unit tangent bundle and it follows that such metrics are clean. We now wish to examine the “cleanliness” of the other naturally reductive metrics on $\text{SO}(3)$. Towards this end we begin by examining the fixed point sets of the geodesic flow for naturally reductive metrics that are not symmetric.

Lemma 4.22. *Consider the naturally reductive metric $g_{(\alpha, \alpha, A)}$ on $U = \text{SO}(3)$, where $\alpha \neq A$, and let $U \times K = \text{SO}(3) \times \text{SO}(2)$ be the connected component of the identity in the isometry group of $g_{(\alpha, \alpha, A)}$. We let $v = c_1 Z_1 + c_2 Z_2 + c_3 Z_3 \in \mathfrak{p} \equiv T_e U$ be a unit vector where $Z_1, Z_2, Z_3 \in T_e U$ is the orthonormal basis given in 4.14(3).*

- (1) *If $c_1^2 + c_2^2 = 1$, then $(U \times K) \cdot v \simeq \text{SO}(3) \times S^1$ and this 4-dimensional submanifold of $T^1 \text{SO}(3)$ accounts for all the unit speed primitive geodesics of Type I, all of which have length $\sqrt{\alpha} \ell_0$. The manifold $(U \times K) \cdot v$ is said to be a Type I component.*
- (2) *If $c_3 = \pm 1$, then $(U \times K) \cdot v \simeq \text{SO}(3)$ and the 3-dimensional submanifold $(U \times K) \cdot v \cup (U \times K) \cdot (-v)$ of $T^1 \text{SO}(3)$ accounts for all the unit speed primitive geodesics of Type II, all of which have length $\sqrt{A} \ell_0$. The manifold $(U \times K) \cdot v$ is said to be a Type II component.*
- (3) *Let $\tau > 0$ be such that $\mathcal{E}_{\tau, \alpha, A}$ is non-empty. For each $(p, q) \in \mathcal{E}_{\tau, \alpha, A}$ fix a unit vector $v_{(p, q)} = c_1 Z_1 + c_2 Z_2 + c_3 Z_3$, where $c_1^2 + c_2^2 = \frac{\sigma(p, q, \alpha, A)}{\sigma(p, q, \alpha, A) + 1}$ and $c_3^2 = \frac{1}{\sigma(p, q, \alpha, A) + 1}$. Then $(U \times K) \cdot v_{(p, q)} \simeq \text{SO}(3) \times S^1_{(p, q)}$, where $S^1_{(p, q)} = \{xZ_1 + yZ_2 + zZ_3 : x^2 + y^2 = \frac{\sigma}{\sigma+1}z = c_3\}$, and the 4-dimensional submanifold $\cup_{(p, q) \in \mathcal{E}_{\tau, \alpha, A}} (U \times K) \cdot (\pm v_{(p, q)})$ accounts for the unit speed primitive geodesics of Type III having length τ . The manifold $(U \times K) \cdot v_{(p, q)}$ is said to be a Type III component.*

Proof. We recall that the isotropy group of the identity element corresponding to the natural action of $U \times K$ on $\text{SO}(3)$ is $\Delta K = \text{SO}(3)$, and as we noted in 4.14(3) the isotropy action of ΔK on $\mathfrak{p} \equiv T_e G$ acts via rotations on $\mathfrak{p}_1 = \text{Span}_{\mathbb{R}}\{Z_1, Z_2\}$ and fixes $\mathfrak{q}_0 = \text{Span}_{\mathbb{R}}\{Z_3\}$. The lemma now follows from Theorem 4.16. \square

Lemma 4.23. *For any $B > 0$, there are finitely many $0 < \tau < B$ such that $\mathcal{E}_{\tau, \alpha, A}$ is non-empty.*

Proof. This follows immediately from the fact that a Type III geodesic has length of the form $\sqrt{\alpha} \ell_0 [q^2 + p^2 \frac{A}{\alpha - A}]^{\frac{1}{2}}$, where $p, q \in \mathbb{N}$, and the values of this function form a discrete subset of \mathbb{R} . \square

Using Theorem 4.16 and Lemmas 4.22 and 4.23 the following is immediate.

Corollary 4.24. *Let $g_{(\alpha, \alpha, A)}$ be a naturally reductive metric on $\mathrm{SO}(3)$ with unit tangent bundle $T^1\mathrm{SO}(3)$ and corresponding geodesic flow $\Phi_t : T^1\mathrm{SO}(3) \rightarrow T^1\mathrm{SO}(3)$, $t \in \mathbb{R}$. Then, for each period τ of the geodesic flow, $\mathrm{Fix}(\Phi_\tau)$ is a union of finitely many (homogeneous) submanifolds of $T^1\mathrm{SO}(3)$ and for each $u \in \mathrm{Fix}(\Phi_\tau)$ the connected component of $\mathrm{Fix}(\Phi_\tau)$ containing u is given by $\mathrm{Isom}(g_{(\alpha, \alpha, A)})^0 \cdot u$, where $\mathrm{Isom}(g_{(\alpha, \alpha, A)})^0$ denotes the connected component of the identity in the isometry group. In particular, we have the following:*

- (1) $\alpha = A$ if and only if $|\tau| = \sqrt{\alpha}\ell_0$ is the length of the shortest non-trivial closed geodesic and $\mathrm{Fix}(\Phi_\tau) = T^1\mathrm{SO}(3)$ is 5-dimensional.
- (2) $A < \alpha$ if and only if $|\tau| = \sqrt{A}\ell_0$ is the length of the shortest non-trivial closed geodesic and $\mathrm{Fix}(\Phi_\tau) \simeq \mathrm{SO}(3) \cup \mathrm{SO}(3)$ is 3-dimensional. In which case all geodesics of length $|\tau| = \sqrt{A}\ell_0$ are of Type II.
- (3) $A > \alpha$ if and only if $|\tau| = \sqrt{\alpha}\ell_0$ is the length of the shortest non-trivial closed geodesic and $\mathrm{Fix}(\Phi_\tau) \simeq \mathrm{SO}(3) \times S^1$ is 4-dimensional. In which case all geodesics of length $|\tau| = \sqrt{\alpha}\ell_0$ are of Type I.

We now give an explicit description of the naturally reductive metrics on $\mathrm{SO}(3)$ which fail to be clean.

Theorem 4.25. *The naturally reductive metric $g_{(\alpha, \alpha, A)}$ is unclean if and only if $A \in \alpha\mathbb{Q}_+ - \{\alpha\}$, where \mathbb{Q}_+ denotes the positive rational numbers. Moreover, if we express $A \in \alpha\mathbb{Q}_+ - \{\alpha\}$ as $A = \frac{2\alpha j}{k}$, where $k, j \in \mathbb{N}$ are relatively prime, then a period τ of the geodesic flow of $g_{(\alpha, \alpha, A)}$ is unclean if and only if $|\tau| = mk\sqrt{A}\ell_0$ for some $m \in \mathbb{N}$.*

Corollary 4.26. *Let $|\tau|$ be the length of the shortest non-trivial closed geodesic with respect to a left-invariant naturally reductive metric on $\mathrm{SO}(3)$. Then, τ is clean.*

Proof. Without loss of generality, we may assume $\tau \geq 0$. Let $g_{(\alpha, \alpha, A)} \in \mathcal{M}_{\mathrm{Nat}}(\mathrm{SO}(3))$ and τ_{\min} denote the length of its shortest non-trivial closed geodesic. If $A \leq \alpha$, then $\tau_{\min} = \sqrt{A}\ell_0$ and in the event that $A > \alpha$ we see that $\tau_{\min} = \sqrt{\alpha}\ell_0$. Now, let $\tau \in \mathrm{Spec}_L^\pm(g_{(\alpha, \alpha, A)})$ be an unclean period of the geodesic flow. Then, by Theorem 4.25, we have that $A \in \alpha\mathbb{Q}_+ - \{\alpha\}$ and, if we express A as $\frac{2j}{k}\alpha$, where j, k are relatively prime, then $|\tau| = mk\sqrt{A}\ell_0$ for some positive integer m . It follows that if $A < \alpha$, then $k \geq 3$ and, therefore, $|\tau| = mk\sqrt{A}\ell_0 > \tau_{\min} = \sqrt{A}\ell_0$. Similarly, if $A > \alpha$, then $|\tau| = mk\sqrt{A}\ell_0 > \tau_{\min} = \sqrt{\alpha}\ell_0$. Therefore, $\tau = \pm\tau_{\min}$ is always clean. \square

Proof of Theorem 4.25. In Corollary 4.24 we have already established that for each τ in the length spectrum of $g_{(\alpha, \alpha, A)}$, the fixed point set $\mathrm{Fix}(\Phi_\tau)$ is the disjoint union of finitely many homogeneous submanifolds $\Theta_1, \dots, \Theta_q$. Hence, our objective is to show that for each $\tau \in \mathrm{Spec}_L(g_{(\alpha, \alpha, A)})$, each $j = 1, \dots, q \equiv q(\tau)$ and each $u \in \Theta_j$ we have

$$(4.27) \quad \ker(D_u\Phi_\tau - \mathrm{Id}_u) = T_u(\Theta_j).$$

That is, we must show that the periodic Jacobi fields $Y(t)$ along the geodesic $\gamma_v(t)$ are precisely those whose initial conditions satisfy $(Y(0), \nabla Y(0)) \in T_v(\Theta_j)$. Since $g_{(\alpha, \alpha, A)}$ is a homogeneous

metric, it is enough to verify this for some $v \in T_e G \cap \text{Fix}(\Phi_\tau)$. And, since the connected components are homogeneous, Corollary 4.11 informs us that $\ker(D_v \Phi_\tau - \text{Id}_v) = T_v(\text{Fix}(\Phi_\tau))$ if and only if $V_4^{\text{per}} = V_4^{\text{iso}}$.

In the case where $A = \alpha$, it is clear that the metric is clean since all geodesics are closed and have the same primitive length ℓ_0 . Therefore, the remainder of our discussion will focus on the case where $A \neq \alpha$.

For any left-invariant naturally reductive metric $g = g_{(\alpha, \alpha, A)}$, where $\alpha \neq A$, and $\tau \in \text{Spec}_L(g_{(\alpha, \alpha, A)})$, if $\Theta \subseteq \text{Fix}(\Phi_\tau)$ is of Type I or III the computations below will show that Equation 4.27 holds for any $u \in \Theta$. However, if Θ is Type II, we will find that Equation 4.27 fails to hold precisely when

- (1) $A = \alpha \frac{2j}{k} \neq \alpha$ with $\gcd(j, k) = 1$, and
- (2) $\tau = mk\sqrt{A}\ell_0$.

We now take up the details.

Suppose that $A \neq \alpha$. Now, let $\mathfrak{p} \equiv T_e G$ denote the $\text{Ad}(\Delta K)$ -invariant complement of $\Delta \mathfrak{K} = \text{Span}\{D\}$ in $\mathfrak{g} \times \mathfrak{K}$. Then, following 4.14(3), the collection $\{Z_1, Z_2, Z_3\}$ forms a g -orthonormal basis for \mathfrak{p} . Hence, any unit vector $v \in \mathfrak{p} \equiv T_e G$ is of the form $c_1 Z_1 + c_2 Z_2 + c_3 Z_3$, where $c_1^2 + c_2^2 + c_3^2 = 1$. By Theorem 4.16 the geodesic $\gamma_v(t) = \exp_{G \times K}(tv) \cdot e$ is closed if and only if one of the following hold:

- (1) $c_1^2 + c_2^2 = 1$ (i.e., γ_v is of Type I);
- (2) $c_3 = \pm 1$ (i.e., γ_v is of Type II);
- (3) $c_1^2 + c_2^2 = \frac{\sigma(p, q, \alpha, A)}{\sigma(p, q, \alpha, A) + 1}$ and $c_3 = \pm \sqrt{\frac{1}{\sigma(p, q, \alpha, A) + 1}}$ for some choice of $p, q \in \mathbb{N}$ relatively prime with $\frac{q^2}{p^2} > (\frac{A}{A-\alpha})^2$ (i.e., γ_v is of Type III).

In the case where γ_v is closed we must determine the fixed point set of the associated Poincaré map $P : E \oplus E \rightarrow E \oplus E$, where (as in Section 4.2) $E = \{u \in \mathfrak{p} : \langle u, v \rangle = 0\}$. By Corollary 4.11, this means we must determine the subspaces $V_1, \dots, V_4^{\text{per}}, V_5 \leq E \oplus E$. In particular, as noted above, we want to determine whether $V_4^{\text{per}} = V_4^{\text{iso}}$. Towards this end, in Figure 1 we have collected information concerning Lie brackets in $\mathfrak{g} \times \mathfrak{K} = \mathfrak{p} \oplus \Delta \mathfrak{K}$ that will be useful in our computations. We now examine the behavior of the Poincaré map associated to the three types of closed geodesics listed above.

Case I: $v = c_1 Z_1 + c_2 Z_2$ with $c_1^2 + c_2^2 = 1$.

By Theorem 4.16 and Corollary 4.24 we see that $v \in \text{Fix}(\Phi_\tau)$ if and only if $\tau = k\sqrt{\alpha}\ell_0$ for $k \in \mathbb{N}$, in which case the connected component of $\text{Fix}(\Phi_\tau)$ containing v is the 4-dimensional manifold $(G \times K) \cdot v \simeq \text{SO}(3) \times S^1$.

Fix $\tau = k\sqrt{\alpha}\ell_0$. Since $v = c_1 Z_1 + c_2 Z_2$ with $c_1^2 + c_2^2 = 1$ we see that $E = \text{Span}\{c_2 Z_1 - c_1 Z_2, Z_3\}$. We now compute the eigenspaces of the self-adjoint map $B : E \rightarrow E$ given by

A	B	$[A, B]_{\Delta \mathfrak{K}}$	$[A, B]_{\mathfrak{p}}$
Z_1	Z_2	$\frac{1}{\sqrt{2}(\bar{A} + \alpha)} D$	$\frac{\sqrt{A}}{\alpha\sqrt{2}} Z_3$
Z_1	Z_3	0	$-\frac{\sqrt{A}}{\alpha\sqrt{2}} Z_2$
Z_2	Z_3	0	$\frac{\sqrt{A}}{\alpha\sqrt{2}} Z_1$
Z_1	D	0	$-\frac{1}{\sqrt{2}} Z_2$
Z_2	D	0	$\frac{1}{\sqrt{2}} Z_1$
Z_3	D	0	0

FIGURE 1. The Lie Bracket in $\mathfrak{g} \times \mathfrak{K} = \mathfrak{p} \oplus \Delta \mathfrak{K}$

$B(\cdot) = -[v, [v, \cdot]_{\Delta \mathfrak{K}}]$. We have

$$\begin{aligned}
 B(Z_3) &= -[c_1 Z_1 + c_2 Z_2, [c_1 Z_1 + c_2 Z_2, Z_3]_{\Delta \mathfrak{K}}] \\
 &= -[c_1 Z_1 + c_2 Z_2, 0] \\
 &= 0
 \end{aligned}$$

and

$$\begin{aligned}
 B(c_2 Z_1 - c_1 Z_2) &= [c_1 Z_1 + c_2 Z_2, [Z_1, Z_2]_{\Delta \mathfrak{K}}] \\
 &= [c_1 Z_1 + c_2 Z_2, \frac{1}{\sqrt{2}(\bar{A} + \alpha)} D] \\
 &= \frac{c_1}{\sqrt{2}(\bar{A} + \alpha)} [Z_1, D] + \frac{c_2}{\sqrt{2}(\bar{A} + \alpha)} [Z_2, D] \\
 &= -\frac{c_1}{\sqrt{2}(\bar{A} + \alpha)} Z_2 + \frac{c_2}{\sqrt{2}(\bar{A} + \alpha)} Z_1 \\
 &= -\frac{1}{\sqrt{2}(\bar{A} + \alpha)} (c_2 Z_1 - c_1 Z_2).
 \end{aligned}$$

Hence, $E_0 = \text{Span}\{Z_3\}$ and $E_1 = \text{Span}\{c_2 Z_1 - c_1 Z_2\}$. Now, let $T : E \rightarrow E$ be the skew-symmetric map $T(\cdot) = -[v, \cdot]_{\mathfrak{p}}$. Then

$$\begin{aligned}
 T(Z_3) &= -c_1 [Z_1, Z_3]_{\mathfrak{p}} - c_2 [Z_2, Z_3]_{\mathfrak{p}} \\
 &= c_1 \frac{\sqrt{A}}{\alpha\sqrt{2}} Z_2 - c_2 \frac{\sqrt{A}}{\alpha\sqrt{2}} Z_1 \\
 &= -\frac{\sqrt{A}}{\alpha\sqrt{2}} (c_2 Z_1 - c_1 Z_2),
 \end{aligned}$$

which is an element of E_1 , and by skew-adjointness we have $T(c_2 Z_1 - c_1 Z_2) = \frac{\sqrt{A}}{\alpha\sqrt{2}} Z_3$ which is an element of E_0 . Therefore, $E_2 = E_0$ and $E_3 = 0$ which implies $E = E_1 \oplus E_2$. We then find that

$$E \oplus E = V_1 \oplus V_2 \oplus V_3 \oplus V_5.$$

In particular, $V_4 = 0$. Consequently, we conclude that the fixed vectors of P coincide with the isotropic Jacobi fields. It then follows that

$$\ker(D_v \Phi_\tau - \text{Id}_v) = T_v \text{Fix}(\Phi_\tau).$$

Case II: $v = \pm Z_3$

By Theorem 4.16 and Corollary 4.24 we see that $v \in \text{Fix}(\Phi_\tau)$ if and only if $\tau = k\sqrt{A}\ell_0$ for $k \in \mathbb{N}$, in which case the connected component of $\text{Fix}(\Phi_\tau)$ containing v is the 3-dimensional manifold $(G \times K) \cdot v \simeq \text{SO}(3)$.

Fix $\tau = k\sqrt{A}\ell_0$, form some $k \in \mathbb{N}$. Since $v = \pm Z_3$, we find that $E = \text{Span}\{Z_1, Z_2\}$. It is then clear that $B \equiv 0$, and we conclude that $E_0 = E$ and $E_1 = 0$. The skew-adjoint map $T : E \rightarrow E$ is given by the following:

$$T(Z_1) = -[Z_3, Z_1]_{\mathfrak{p}} = -\frac{\sqrt{A}}{\alpha\sqrt{2}}Z_2$$

and

$$T(Z_2) = -[Z_3, Z_2]_{\mathfrak{p}} = \frac{\sqrt{A}}{\alpha\sqrt{2}}Z_1.$$

Hence, $E_2 \equiv \{\Theta \in E_0 : T(\Theta) \in E_1\} = 0$, $E_3 = E_0 = E$ and we conclude that

$$E \oplus E = V_1 \oplus V_4.$$

Therefore, since V_1 is 2-dimensional and the connected component of $\text{Fix}(\Phi_\tau)$ containing v is 3-dimensional we see that $T_v(\text{Fix}(\Phi_\tau)) = V_1 \oplus \text{Span}\{(v, 0)\}$, which implies $V_4^{\text{iso}} = 0$. This last equality can also be seen by recalling that $(X, \frac{1}{2}[v, X]_{\mathfrak{p}}) \in V_4$ gives rise to a non-trivial isotropic Jacobi field along γ_v if and only if $X \neq 0 \in E_3$ is such that $T(X) \in [\Delta\mathfrak{K}, v]$. However, since $[\Delta\mathfrak{K}, v] = 0$ and $T : E \rightarrow E$ is an isomorphism, no such vector exists and we see that $V_4^{\text{iso}} = 0$. Hence, if P has non-trivial fixed vectors in V_4 (i.e., $V_4^{\text{per}} \neq 0$), they will not lie in $T_v(\text{Fix}(\Phi_\tau))$.

We now recall that $(X, \frac{1}{2}[v, X]_{\mathfrak{p}}) \in V_4$ is fixed by P if and only if $\Psi(X) = X$, where $\Psi : E \rightarrow E$ is given by $\Psi = e^{\text{ad}(k\sqrt{A}\ell_0 v)}$. Now, since Z_1 and Z_2 span E and $v = Z_3$, it follows that $\text{ad } v = -T$; therefore,

$$\Psi = e^{-k\sqrt{A}\ell_0 T}.$$

With respect to the basis $\{Z_1, Z_2\}$ of E we see that $-k\sqrt{A}\ell_0 T$ is represented by the following matrix

$$\begin{pmatrix} 0 & -\theta(\alpha, A) \\ \theta(\alpha, A) & 0 \end{pmatrix},$$

where $\theta(\alpha, A) = \frac{kA\ell_0}{\alpha\sqrt{2}} = \frac{kA\pi}{\alpha}$. Hence, with respect to the basis $\{Z_1, Z_2\}$, Ψ has the following matrix

$$\begin{pmatrix} \cos \theta(\alpha, A) & \sin \theta(\alpha, A) \\ -\sin \theta(\alpha, A) & \cos \theta(\alpha, A) \end{pmatrix}.$$

Therefore, Ψ has a fixed vector if and only if $\theta(\alpha, A) \in 2\pi\mathbb{N}$, which is equivalent to $A \in \frac{2\alpha}{k}\mathbb{N}$. This implies that $\ker(D_v \Phi_\tau - \text{Id}_v) \neq T_v(\text{Fix}(\Phi_\tau))$ if and only if $A \in \frac{2\alpha}{k}\mathbb{N}$. Since $k \in \mathbb{N}$

is arbitrary, we may conclude that in the case where $A \neq \alpha$ we have $\ker(D_v \Phi_\tau - \text{Id}_v) \neq T_v(\text{Fix}(\Phi_\tau))$ if and only if $A = \frac{2\alpha j}{k}$, where j and k are relatively prime, and $\tau = m(k\sqrt{A}\ell_0)$ for some $m \in \mathbb{N}$.

Case III: $v = c_1 Z_1 + c_2 Z_2 + c_3 Z_3$, where $c_1^2 + c_2^2 = \frac{\sigma(p,q,\alpha,A)}{1+\sigma(p,q,\alpha,A)}$ and $c_3 = \pm \sqrt{\frac{1}{1+\sigma(p,q,\alpha,A)}}$ for unique $p, q \in \mathbb{N}$ relatively prime such that $\frac{q^2}{p^2} > (\frac{A}{\alpha-A})^2$.

By Theorem 4.16 and Corollary 4.24, we see that in this case $v \in \text{Fix}(\Phi_\tau)$ if and only if $\tau = k\sqrt{\alpha}\ell_0(q^2 + p^2 \frac{A}{(\alpha-A)})^{\frac{1}{2}}$ for $k \in \mathbb{N}$, in which case the connected component of $\text{Fix}(\Phi_\tau)$ containing v is the 4-dimensional manifold $(G \times K) \cdot v \simeq \text{SO}(3) \times S^1$.

$\text{Fix } \tau = k\sqrt{\alpha}\ell_0(q^2 + p^2 \frac{A}{(\alpha-A)})^{\frac{1}{2}}$ for some $k \in \mathbb{N}$ and notice that $E = \text{Span}\{c_2 Z_1 - c_1 Z_2, c_1 c_3 Z_1 + c_2 c_3 Z_2 - (c_1^2 + c_2^2) Z_3\}$. To find the eigenspaces of $B : E \rightarrow E$ we observe that

$$\begin{aligned} B(c_2 Z_1 - c_1 Z_2) &= -[v, [v, c_2 Z_1 - c_1 Z_2]_{\Delta \mathfrak{K}}] \\ &= -[c_1 Z_1 + c_2 Z_2 + c_3 Z_3, -(c_1^2 + c_2^2)[Z_1, Z_2]_{\Delta \mathfrak{K}}] \\ &= -[c_1 Z_1 c_2 Z_2 + c_3 Z_3, -\frac{(c_1^2 + c_2^2)}{\sqrt{2}(\bar{A} + \alpha)} D] \\ &= \frac{(c_1^2 + c_2^2)}{\sqrt{2}(\bar{A} + \alpha)} (c_1 [Z_1, D] + c_2 [Z_2, D] + c_3 [Z_3, D]) \\ &= \frac{(c_1^2 + c_2^2)}{\sqrt{2}(\bar{A} + \alpha)} (c_2 Z_1 - c_1 Z_2) \end{aligned}$$

and

$$\begin{aligned} B(c_1 c_3 Z_1 + c_2 c_3 Z_2 - (c_1^2 + c_2^2) Z_3) &= -[v, [v, c_1 c_3 Z_1 + c_2 c_3 Z_2 - (c_1^2 + c_2^2) Z_3]_{\Delta \mathfrak{K}}] \\ &= -[c_1 Z_1 + c_2 Z_2 + c_3 Z_3, c_1 c_2 c_3 [Z_1, Z_2]_{\Delta \mathfrak{K}} - c_1 c_2 c_3 [Z_1, Z_2]_{\Delta \mathfrak{K}}] \\ &= 0. \end{aligned}$$

Hence, $E_0 = \text{Span}\{c_1 c_3 Z_1 + c_2 c_3 Z_2 - (c_1^2 + c_2^2) Z_3\}$ and $E_1 = \text{Span}\{c_2 Z_1 - c_1 Z_2\}$. We now determine E_2 and E_3 by computing $T : E \rightarrow E$:

$$\begin{aligned} T(c_1 c_3 Z_1 + c_2 c_3 Z_2 - (c_1^2 + c_2^2) Z_3) &= -[c_1 Z_1 + c_2 Z_2 + c_3 Z_3, c_1 c_3 Z_1 + c_2 c_3 Z_2 - (c_1^2 + c_2^2) Z_3]_{\mathfrak{p}} \\ &= c_1 [Z_1, Z_3]_{\mathfrak{p}} + c_2 [Z_2, Z_3]_{\mathfrak{p}} \\ &= \frac{\sqrt{A}}{\alpha\sqrt{2}} (c_2 Z_1 - c_1 Z_2) \end{aligned}$$

and we also see that

$$T(c_2 Z_1 - c_1 Z_2) = -\frac{\sqrt{A}}{\alpha\sqrt{2}} (c_1 c_3 Z_1 + c_2 c_3 Z_2 - (c_1^2 + c_2^2) Z_3).$$

It follows that $E_2 = \{X \in E_0 : T(X) \in E_1\} = E_0$ and $E_3 = 0$, which allows us to see that $E = E_1 \oplus E_2$. Therefore, $V_4 = 0$ and

$$E \oplus E = V_1 \oplus V_2 \oplus V_3 \oplus V_5.$$

Hence, the only fixed vectors of P come from isotropic Jacobi fields and we have

$$\ker(D_v\Phi_\tau - I_v) = T_v(\text{Fix}(\Phi_\tau)).$$

Cases I - III now clearly imply the theorem. Indeed, when $\alpha \neq A$, we see that the cleanliness of $\tau \in \text{Spec}_L(g_{(\alpha, \alpha, A)})$ hinges on the behavior of the Poincaré map along geodesics of length τ having Type II. The conclusion of Case II, then gives us the main statement of the theorem. \square

Remark 4.28. It is clear from the proof of Theorem 4.25 that the cleanliness of a metric is dictated by the behavior of the Poincaré map along Type II geodesics.

Proof of Theorem 1.9. The space of naturally reductive left-invariant metrics on $\text{SO}(3)$ is identified with $\mathcal{A} = \{(\alpha, A) : \alpha, A > 0\} \subset \mathbb{R}^2$. Now for each $r \in \mathbb{Q}_+$ let $\mathcal{A}_r \equiv \{(\alpha, A) : A = r\alpha\}$. Then it follows from Theorem 4.25 that the class of clean metrics in \mathcal{A} is given by $\mathcal{C} = \bigcap_{r \neq 1 \in \mathbb{Q}_+} (\mathcal{A} - \mathcal{A}_r)$, which is a residual set containing the bi-invariant metrics \mathcal{A}_1 .

The final statement follows from the fact that the normal homogeneous metrics on $\text{SO}(3)$ are identified with the set $\mathcal{N} = \{(\alpha, A) : \alpha \leq A\}$ and, by Theorem 4.25, we see that $\mathcal{N} \cap (\mathcal{A} - \mathcal{C}) \neq \emptyset$. \square

4.4. The 0-th Wave Invariant and the Poisson Relation. Is the Poisson relation an equality for the clean left-invariant naturally reductive metrics on $\text{SO}(3)$? In light of the argument employed in Theorem 1.13, it is reasonable to wonder whether one can show that for such metrics the leading term of the trace formula is non-zero for each period τ . We will observe that although this is clear when $|\tau|$ is the length of the shortest non-trivial geodesic, this does not appear to be an easy matter to resolve, in general. We begin by reviewing the construction of the Duistermaat-Guillemin measure as discussed in the appendix of [BPU].

Constructing the Duistermaat-Guillemin Measure. Let τ be a clean period of the geodesic flow. For simplicity we will assume that $\Theta = \text{Fix}(\Phi_\tau)$ is connected and we will let $\tilde{\Theta} = \{tX_p : X_p \in F \text{ and } t > 0\}$. We will exploit the symplectic structure of the tangent bundle to construct a canonical measure $\tilde{\mu}^\tau$ on $\tilde{\Theta}$ and obtain a canonical measure on Θ by dividing by the measure $|dq|$ (in the transverse direction).

Indeed, one can check that $\tilde{\Theta}$ is a clean fixed point set of $\tilde{\Phi}_\tau$. Now, let $z \in \tilde{\Theta}$ and consider $T = Id_z - D_z\Phi_\tau : V \rightarrow V$, where $V \equiv T_z TM$. Following [BPU, p. 524-525] we can construct a density on $T_z\tilde{\Theta}$ as follows.

- Let $\mathcal{E} = \{e_1, \dots, e_k\}$ be a basis for $W \equiv T_z\tilde{\Theta}$;
- Let $W^\Omega = \{v \in V : \Omega(w, v) = 0 \text{ for each } w \in W\}$ be the Ω -orthogonal complement of W in V .
- Let $\mathcal{F} = \{f_1, f_2, \dots, f_k\}$ be a basis for a complement of W^Ω satisfying

$$\Omega(e_i, f_j) = \delta_{ij}.$$

- Let $\mathcal{V} = \{v_1, \dots, v_{2n-k}\}$ be a basis for a complement of W in V .

With the above notation we have the following lemma.

Lemma 4.29 (Lemma A.2 [BPU]).

- (1) $\ker(T) = W$ and the image of T equals W^Ω , so that $T\mathcal{V} \cup \mathcal{F}$ is a basis for V .
- (2) Let $\varphi \in |V|^{1/2}$ be an arbitrary half-density on V . Then the DG-density $\tilde{\mu}^\tau$ on $W \equiv T_z \tilde{\Theta}$ is given by

$$\tilde{\mu}^\tau(\mathcal{E}) = \frac{\varphi(\mathcal{V} \wedge \mathcal{E})}{\varphi(T\mathcal{V} \wedge \mathcal{F})} = \frac{1}{|\alpha(u)|^{1/2}},$$

where we abuse notation and have $\mathcal{E} = e_1 \wedge \cdots \wedge e_k$, $\mathcal{F} = f_1 \wedge \cdots \wedge f_k$, $\mathcal{V} = v_1 \wedge \cdots \wedge v_{2n-k}$, $T\mathcal{V} = Tv_1 \wedge \cdots \wedge Tv_{2n-k}$ and $\alpha(u) \neq 0$ satisfies $T\mathcal{V} \wedge \mathcal{F} = \alpha(z)\mathcal{V} \wedge \mathcal{E}$.

It then follows that if we let $\nu_{\tilde{g}|_\Theta}$ denote the Riemannian density on Θ induced by the Sasaki metric \tilde{g} on TM associated to g , then the Duistermaat-Guillemin measure μ^τ on Θ is given by

$$\mu^\tau = \frac{1}{|\alpha|^{1/2}} \nu_{\tilde{g}|_\Theta},$$

where for each $z \in \Theta$ the function $\alpha(z)$ is computed as in the preceding lemma.

Using the procedure outlined above we obtain the following.

Lemma 4.30. Let $g = g_{(\alpha, \alpha, A)}$ be a left-invariant naturally reductive metric on $\mathrm{SO}(3)$ with corresponding Sasaki metric \tilde{g} on the tangent bundle. Let τ be a clean period of the geodesic flow in the length spectrum of g and $d\mu^\tau$ denote the corresponding DG-measure on $\mathrm{Fix}(\Phi_\tau)$ (see p. 6). And, the set $\mathcal{E}_{\tau, \alpha, A}$ and function $\sigma(p, q, \alpha, A)$ are as in Theorem 4.16.

- (1) If $\alpha = A$, then $\mathrm{Fix}(\Phi_{\tau_{\min}}) = T^1 \mathrm{SO}(3) = \mathrm{SO}(3) \times S_1^2$ and $d\mu^\tau = d\nu_{\tilde{g}|_{\mathrm{Fix}(\Phi_\tau)}}$. That is, $d\mu^\tau$ is the Riemannian density on $\mathrm{Fix}(\Phi_\tau)$ induced by the restriction of the Sasaki metric. And, we have

$$\int_{\mathrm{Fix}(\Phi_\tau)} d\mu^\tau = \mathrm{vol}(g) \cdot \mathrm{vol}(S_1^2) = 4\pi \mathrm{vol}(g)$$

- (2) For $\alpha \neq A$ the components of $\mathrm{Fix}(\Phi_\tau)$ are of Type I, II or III (see Lemma 4.22).
 - (a) Suppose $\Theta \subset \mathrm{Fix}(\Phi_\tau)$ is a component of Type I. Then, the restriction of the DG-measure to Θ is given by $d\mu^\tau \upharpoonright \Theta \equiv \frac{1}{\sqrt{|\tau|}} d\nu_{\tilde{g}|_\Theta}$ and

$$\int_\Theta d\mu^\tau = \frac{1}{\sqrt{|\tau|}} \mathrm{vol}(g) \mathrm{vol}(S_1^1) = \frac{2\pi}{\sqrt{|\tau|}} \mathrm{vol}(g).$$

- (b) Suppose $\Theta \subset \mathrm{Fix}(\Phi_\tau)$ is a component of Type II. Then, the restriction of the DG-measure to Θ is given by $d\mu^\tau \upharpoonright \Theta \equiv \frac{1}{|\tau|} d\nu_{\tilde{g}|_\Theta}$ and

$$\int_\Theta d\mu^\tau = \frac{1}{|\tau|} \mathrm{vol}(g).$$

- (c) Suppose $\Theta \subset \text{Fix}(\Phi_\tau)$ is a component of Type III, so that $\Theta = (G \times K) \cdot v_{(p,q)}$ for $(p, q) \in \mathcal{E}_{\tau, \alpha, A}$ and $v_{(p,q)} \in T_e G$ as in Lemma 4.22(3). Then, the restriction of the DG-measure to Θ is given by $d\mu^\tau \upharpoonright \Theta \equiv \frac{1}{\sqrt{|\tau|}} d\nu_{\tilde{g}|_\Theta}$ and

$$\int_{\Theta} d\mu^\tau = \frac{2\pi}{\sqrt{|\tau|}} \sqrt{\frac{\sigma(p, q, \alpha, A)}{\sigma(p, q, \alpha, A) + 1}} \text{vol}(g).$$

As an application we obtain the wave invariant associated to the systole of a left-invariant naturally reductive metric on $\text{SO}(3)$. The fact that these wave-invariants are non-zero, establishes the “audibility” of the systole in this setting.

Lemma 4.31. *Let $g = g_{(\alpha, \alpha, A)}$ be a left-invariant naturally reductive metric on $\text{SO}(3)$. Let $\tau_{\min} = \tau_{\min}(g)$ denote the length of the shortest closed geodesic with respect to g and σ denote the Morse index of any smooth closed geodesic with respect to g having length τ_{\min} .*

- (1) *If $\alpha = A$, then $\tau_{\min} = \sqrt{\alpha} \ell_0$ is clean, $\text{Fix}(\Phi_{\tau_{\min}})$ is a connected manifold of dimension 5, and all of the closed geodesics of length τ_{\min} have Morse index $\sigma = 0$. Furthermore,*

$$\text{Wave}_0^{\text{odd}}(\pm \tau_{\min}) = -\frac{1}{\pi} \text{vol}(g)$$

and $\text{Wave}_k^{\text{even}}(\pm \tau_{\min}) = 0$ for $k \geq 0$.

- (2) *If $A < \alpha$, then $\tau_{\min} = \sqrt{A} \ell_0$ is clean, $\text{Fix}(\Phi_{\tau_{\min}})$ is a manifold of dimension 3 having two connected components, and the closed geodesics of length τ_{\min} all have the same Morse index σ . Furthermore,*

$$\text{Wave}_0^{\text{odd}}(\pm \tau_{\min}) = \frac{i^{-(\sigma+1)} \text{vol}(g)}{\pi \tau_{\min}}.$$

$\text{Wave}_k^{\text{even}}(\pm \tau_{\min}) = 0$ for $k \geq 0$.

- (3) *If $A > \alpha$, then $\tau_{\min} = \sqrt{\alpha} \ell_0$ is clean, $\text{Fix}(\Phi_{\tau_{\min}})$ is a connected manifold of dimension 4, and the closed geodesics of length τ_{\min} all have common Morse index σ . Furthermore,*

$$\text{Wave}_0^{\text{even}}(\pm \tau_{\min}) = \left(\frac{1}{2\pi i} \right)^{3/2} i^{-\sigma} \frac{2\pi}{\sqrt{\tau_{\min}}} \text{vol}(g).$$

$\text{Wave}_k^{\text{odd}}(\pm \tau_{\min}) = 0$ for $k \geq 0$.

Consequently, for each left-invariant naturally reductive metric on $\text{SO}(3)$, τ_{\min} can be recovered from its spectrum.

Remark 4.32. It is possible to use the 0-th wave-invariant associated to $\tau_{\min}(g)$ to demonstrate that the left-invariant naturally reductive metrics on $\text{SO}(3)$ can be mutually distinguished via their spectra. In terms of physical chemistry this means that for a spherical or symmetric molecule (e.g., methane, benzene or chloromethane) its moments of inertia can be recovered from its rotational spectrum. For a more general discussion see [SS], where we use the heat trace to prove the following more general statement: If (M_1, g_1) and (M_2, g_2) form a non-trivial isospectral pair of homogeneous three-manifolds, then M_1 and M_2 are both spherical

three-manifolds with non-isomorphic fundamental groups and equipped with Type I metrics. Consequently, the moments of inertia of any molecule can be recovered from its rotational spectrum.

Let $g_{(\alpha,\alpha,A)}$ be a left-invariant naturally reductive metric on $U = \text{SO}(3)$ with $\alpha \neq A$. Fix an element $\tau \in \text{Spec}_L^\pm(g_{(\alpha,\alpha,A)})$ and recall that $\text{Fix}(\Phi_\tau)$ consists of components of Type I, II and III. We observe that $\text{Fix}(\Phi_\tau)$ cannot contain components of Type I and Type II simultaneously. Indeed, if this were the case, then we could find natural numbers m and n such that $|\tau| = m\sqrt{\alpha}\ell_0 = n\sqrt{A}\ell_0$, which would imply that $A \in \alpha\mathbb{Q}_+ - \{\alpha\}$, contradicting the fact that the metric $g_{(\alpha,\alpha,A)}$ is clean (see Theorem 4.25). It is also the case that Type I components cannot occur along with Type III components. For otherwise, there exist natural numbers m and n such that $|\tau| = m\sqrt{\alpha}\ell_0 = n\sqrt{A}\ell_0(q^2 + p^2 \frac{A}{\alpha-A})^{1/2}$, which implies that $A \in \alpha\mathbb{Q}_+ - \{\alpha\}$ and leads us to conclude that the metric $g_{(\alpha,\alpha,A)}$ is actually unclean, which is a contradiction.

Now, let $|\tau| = n\sqrt{\alpha}\ell_0$ be the length of a Type I geodesic. Then, the previous paragraph dictates that $\text{Fix}(\Phi_\tau)$ consists of the lone Type I component. Therefore, since the Type I component is of dimension 4 we see that $\text{Wave}_0^{\text{even}}(\tau) \neq 0$. Therefore, the length of any Type I geodesic is contained in the singular support of the trace of the wave group.

To analyze periods arising from Type II and Type III geodesics, we recall that the Type II components are 3-dimensional while the Type III components are 4-dimensional. Hence, given a period τ not arising from a Type I geodesic, the Type II components determine $\text{Wave}_k^{\text{odd}}(\tau)$ and Type III components determine $\text{Wave}_k^{\text{even}}(\tau)$.

Let τ be the period of a Type II orbit of the geodesic flow, then the odd-dimensional part of $\text{Fix}(\Phi_\tau)$ is of the form $(U \times K) \cdot v \cup (U \times K) \cdot -v$, where v is the initial velocity of some unit speed geodesic of Type II (see Lemma 4.22). Then, since the Morse index associated to these components is clearly the same, we conclude that $\text{Wave}_0^{\text{odd}}(\tau)$ is non-zero and, therefore, τ is also in the singular support of the trace of the wave group.

Now, let τ be the period of a Type III orbit of the geodesic flow, then the even-dimensional part of $\text{Fix}(\Phi_\tau)$ is a finite union of Type III components of dimension 4: each component is of the form $(U \times K) \cdot v_{(p,q)} \simeq \text{SO}(3) \times S^1$ (see Lemma 4.22). Using Ziller's recasting of the Jacobi equation for naturally reductive metrics [Z2] and our explicit understanding of the Poincaré map, one can show that the conjugate points along a Type III geodesic are as follows.

Proposition 4.33. *Suppose $\gamma_{v_{(p,q)}}$ is a Type III geodesic with $v_{(p,q)} = c_1Z_1 + c_2Z_2 + c_3Z_3 \in \mathfrak{p} \equiv T_eU$, where $c_1^2 + c_2^2 = \frac{\sigma(p,q,\alpha,A)}{\sigma(p,q,\alpha,A)+1}$ and $c_3 = \pm \sqrt{\frac{1}{\sigma(p,q,\alpha,A)+1}}$ for a unique $(p,q) \in \mathcal{E}_{\tau,\alpha,A}$, where $\sigma(p,q,\alpha,A)$ and $\mathcal{E}_{\tau,\alpha,A}$ are defined as in Theorem 4.16. And, let*

$$a(p,q,\alpha,A) = \sqrt{\varphi^2 + \frac{\sigma(p,q,\alpha,A)}{\sigma(p,q,\alpha,A)+1} \frac{1}{2(\bar{A} + \alpha)}},$$

where $\varphi \equiv \frac{1}{\alpha}\sqrt{\frac{A}{2}}$ and $\bar{A} \equiv \frac{A\alpha}{\alpha-A}$.

- (1) If $\alpha > A$, then $\gamma_{v_{(p,q)}}(t_0)$, $t_0 > 0$, is conjugate to $e = \gamma_{v_{(p,q)}}(0)$ along $\gamma_{v_{(p,q)}}$ if and only if $t_0 \neq 0 \in \frac{2\pi}{a(p,q,\alpha,A)}\mathbb{N}$. And, in this case the conjugate point has multiplicity one.
- (2) If $\alpha < A$, then $\gamma_{v_{(p,q)}}(t_0)$, $t_0 > 0$, is conjugate to $e = \gamma_{v_{(p,q)}}(0)$ along $\gamma_{v_{(p,q)}}$ if and only if $t_0 \neq 0 \in \frac{2\pi}{a(p,q,\alpha,A)}\mathbb{N}$ or $t_0 = \frac{4\alpha^2}{(A-\alpha)\frac{\sigma(p,q,\alpha,A)}{\sigma(p,q,\alpha,A)+1}}$. And, in this case the conjugate point has multiplicity one.

Using the previous proposition one can compute the Morse index associated to each Type III component $(U \times K) \cdot v_{(p,q)}$ contained in $\text{Fix}(\Phi_\tau)$. This, in conjunction with computations in the spirit of those used to establish Lemma 4.30(2c), allows one to compute the contribution of each Type III component to the wave invariant $\text{Wave}_0^{\text{odd}}(\tau)$. However, some inspection will demonstrate that these contributions behave rather erratically making it difficult to rule out the possibility of cancellations, in general. Therefore, at the moment, the best we can say is the following.

Proposition 4.34. *Let $g = g_{(\alpha,\alpha,A)}$ be a left-invariant naturally reductive metric on $\text{SO}(3)$, then*

$$\{\pm n\sqrt{\alpha}\ell_0, \pm n'\sqrt{\alpha}\ell_0 : n, n' \geq 0\} \subseteq \text{SingSupp}(\text{Tr}(e^{-it\sqrt{\Delta_g}})) \subseteq \text{Spec}_L^\pm(g).$$

In particular, $\tau_{\min}(g)$ can be recovered from the Laplace spectrum.

Consequently, it is unclear whether the wave group of a left-invariant naturally reductive metric on $\text{SO}(3)$ is singular at periods of the geodesic flow that can only arise from Type III orbits.

APPENDIX A. ABSTRACT ROOT SYSTEMS

We compute the co-root lattice, central lattice and lowest dominant forms associated to the indecomposable root systems. We also compute the integral lattice associated to each simple Lie group. This data is used in the proof of Theorem 1.23 in Section 3. References for some aspects of this section include [Hel, Chp. 10.3], [BtD, Chp. V.6], [Lo, Chp. 7.3], [Sa, Chp. 2.14], [Hum, Chp. III] and [Ad].

A.1. Basic Definitions. Let V be a finite dimensional vector space over \mathbb{R} and let V^* denote its dual space. A *root system* associated to V is a finite subset $R \subset V^*$ having the following properties

- (1) R spans V^* ;
- (2) There is a map $\lambda \mapsto \lambda^\vee$ from R to V such that for any $\alpha, \beta \in R$ we have
 - (a) $\alpha(\alpha^\vee) = 2$
 - (b) $\beta(\alpha^\vee) \in \mathbb{Z}$;
 - (c) $\beta - \beta(\alpha^\vee)\alpha \in R$

In the event that $c\alpha \in R$ implies $c = \pm 1$ for any choice of $c \in \mathbb{R}, \alpha \in R$ we say that the root system is *reduced*; otherwise, we say the root system is *non-reduced* in which case c is restricted to the values $\pm\frac{1}{2}, \pm 1, \pm 2$. The dimension of V is the *rank* of the root system. For

each $\alpha \in R$, α^\vee is called the *co-root* of α and the set $R^\vee \subset V \simeq V^{**}$ consisting of the co-roots is a root system associated to V^* known as the *co-root system*.

Associated to each $\alpha \in R$ we have the *reflection through the root α* which is defined to be the isomorphism $S_\alpha : V \rightarrow V$ given by $S_\alpha(X) = X - \alpha(X)\alpha^\vee$. The map S_α fixes the hyperplane $\ker(\alpha)$ and sends α^\vee to $-\alpha^\vee$. The *Weyl group* associated to R is the finite group W generated by the S_α 's. If for any $\lambda \in V^*$ we define $S_\alpha \cdot \lambda \equiv \lambda \circ S_\alpha^{-1} = \lambda \circ S_\alpha$, then it follows from condition 2(c) that W acts on R via permutations. We also recall that if $\langle \cdot, \cdot \rangle$ is a W -invariant inner product on V and if for each $\lambda \in V^*$ we let $\lambda^* \in V$ be the unique vector such that $\langle \lambda^*, \cdot \rangle = \lambda(\cdot)$, then $\alpha^\vee = \frac{2\alpha^*}{\langle \alpha^*, \alpha^* \rangle}$. It follows that for any $\beta, \alpha \in R$ we have $\beta(\alpha^\vee) = 0$ if and only if $\langle \beta^*, \alpha^* \rangle = 0$, we therefore agree to say that two roots $\alpha, \beta \in R$ are *orthogonal* if $\beta(\alpha^\vee) = 0$.

A vector $v \in V$ is said to be *regular* if $\alpha(v) \neq 0$ for every $\alpha \in R$; otherwise, we will say the vector is *singular*. The *Weyl chambers* associated to the root system R are the connected components of $V - \bigcup_{\alpha \in R} \ker(\alpha)$ and a choice of Weyl chamber \mathcal{C} allows us to partition R into two disjoint sets:

$$R^+ = R^+(\mathcal{C}) \equiv \{\alpha \in R : \alpha \upharpoonright \mathcal{C} > 0\}$$

and

$$R^- = R^-(\mathcal{C}) \equiv \{\alpha \in R : \alpha \upharpoonright \mathcal{C} < 0\}.$$

The roots in R^+ (resp. R^-) are known as the *positive* (resp. *negative*) *roots* of R with respect to \mathcal{C} . An element $\alpha \in R^+$ is said to be *decomposable* if there are $\alpha_1, \alpha_2 \in R^+$ such that $\alpha = \alpha_1 + \alpha_2$; otherwise, we say that $\alpha \in R^+$ is *indecomposable*.

Definition A.1. Let R be a root system and \mathcal{C} an associated Weyl chamber. A subset $B \subset R^+ = R^+(\mathcal{C})$ is said to be a (*positive*) *basis* for R if the following conditions are met:

- (1) B is a vector space basis of V^* ;
- (2) The coordinates of each root $\alpha \in R$ with respect to the basis B are all non-negative integers or all non-positive integers.

The elements of B are called *simple roots*.

Theorem A.2 ([Hum] Sec. 10). *Every root system has a positive basis. In fact, if R is a root system and \mathcal{C} is an associated Weyl chamber, the set $B^+(\mathcal{C})$ consisting of the indecomposable elements in $R^+(\mathcal{C})$ forms a positive basis and all positive bases of R arise in this manner.*

Corollary A.3. *Let B be a positive basis of the root system R , then the set $B^\vee = \{\alpha^\vee : \alpha \in B\}$ is a positive basis for R^\vee .*

Associated to any root system $R \subset V^*$ there are two important lattices in V .

Definition A.4. Let V be a vector space and $R \subset V^*$ a root system with co-root system $R^\vee \subset V$.

- (1) The *co-root lattice* associated to R is the lattice Λ_{R^\vee} generated by the co-roots.
- (2) The *central lattice* associated to R is the lattice $\Lambda_Z = \{v \in V : \alpha(v) \in \mathbb{Z} \text{ for all } \alpha \in R\}$.

Clearly, $\Lambda_{R^\vee} \subseteq \Lambda_Z$ and in light of Corollary A.3 we have $\Lambda_{R^\vee} = \langle B^\vee \rangle$, for any positive basis B .

Given a root system $R \subset V^*$ of V with positive roots R^+ corresponding to some choice of Weyl chamber, the *lowest strongly dominant form* is the element $\rho \equiv \frac{1}{2} \sum_{\alpha \in R^+} \alpha$ (i.e., *half* the sum of the positive roots). This element enjoys the following well-known property which we will find useful in our proof of Proposition 1.18.

Lemma A.5. *ρ is integer valued on the co-root lattice. In fact, ρ assumes the value 1 on each element of B^\vee .*

Proof. Let B be a basis for R . Then, since B^\vee is a basis for R^\vee , we only need to check the value of ρ on B^\vee . Now, we recall that for any $\alpha \in R$, the reflection S_α permutes the elements in $R^+ - \{\alpha\}$ and sends α to $-\alpha$. Therefore, $S_\alpha \cdot \rho = \rho - \alpha$. On the other hand, by definition, $S_\alpha \cdot \rho = \rho - \rho(\alpha^\vee)\alpha$. Therefore, since B is a basis, we conclude that $\rho(\alpha^\vee) = 1$. \square

A.2. Classification of Indecomposable Root Systems. Given a vector space V a root system $R \subset V^*$ is said to be *decomposable* if it can be written as the disjoint union of two non-empty, orthogonal sets R_1 and R_2 ; otherwise, we will say that R is *indecomposable*. In the event that R is decomposable with orthogonal decomposition $R = R_1 \cup R_2$, if we let V_j^* be the span of R_j , for $j = 1, 2$, then $V^* = V_1^* \oplus V_2^*$, where $V_j^* = \text{Span}_{\mathbb{R}}(R_j)$ for $j = 1, 2$, and one easily sees that R_j is a root system of V_j^* , for $j = 1, 2$. The indecomposable root systems have been classified up to isomorphism. As it will prove useful in Section 3 and all of this information does not appear in one convenient location in the literature, we now give an explicit description of these root systems along with their co-root lattices, central lattices, center and lowest strongly dominant forms. Throughout we will let e_1, \dots, e_n be the standard basis for \mathbb{R}^n and $\epsilon_1, \dots, \epsilon_n$ will denote the standard dual basis: $\epsilon_k(e_j) = \delta_{jk}$. We also let $\langle \cdot, \cdot \rangle$ denote the standard inner product on \mathbb{R}^n , so that $\epsilon_j^* = e_j$ for $j = 1, \dots, n$. In all cases our vector space V will be a subspace of the inner product space $(\mathbb{R}^n, \langle \cdot, \cdot \rangle)$ and $\langle \cdot, \cdot \rangle$ will be invariant under the action of the Weyl group.

A.2.1. Type A_n . As a vector space we have $V = \{v \in \mathbb{R}^{n+1} : \sum_{j=1}^{n+1} \epsilon_j(v) = 0\}$. The corresponding root system is $R = \{(\epsilon_\mu - \epsilon_\nu) \upharpoonright V : 1 \leq \mu \neq \nu \leq n+1\}$ with co-root system $R^\vee = \{e_\mu - e_\nu : 1 \leq \mu \neq \nu \leq n+1\}$. For a Weyl chamber we choose $\mathcal{C} = \{v \in V : \epsilon_\nu(v) > \epsilon_{\nu+1}(v), 1 \leq \nu \leq n\}$, which gives the corresponding positive roots $R^+ = \{(\epsilon_\mu - \epsilon_\nu) \upharpoonright V : 1 \leq \mu < \nu \leq n+1\}$ and positive basis $B = \{(\epsilon_\nu - \epsilon_{\nu+1}) \upharpoonright V : 1 \leq \nu \leq n\}$. The co-root lattice, central lattice, center and sum of the positive roots (i.e., 2ρ) of this root system are given by

- $\Lambda_{R^\vee} = \langle e_\mu - e_\nu : 1 \leq \mu \neq \nu \leq n+1 \rangle$
- $\Lambda_Z = \langle L_j \equiv \frac{n}{n+1}e_j - \frac{1}{n+1} \sum_{\substack{k=1 \\ k \neq j}}^{n+1} e_k : 1 \leq j \leq n \rangle = \langle L_1, e_\mu - e_\nu : 1 \leq \mu < \nu \leq n+1 \rangle$
- $Z = \Lambda_Z / \Lambda_{R^\vee} = \langle \bar{L}_1 \rangle \simeq \mathbb{Z}_{n+1}$
- $2\rho \equiv \sum_{\alpha \in R^+} \alpha = \sum_{\mu=1}^n (2n - 2\mu + 2)\epsilon_\mu$, since $\epsilon_{n+1} = -\sum_{\mu=1}^n \epsilon_\mu$

The corresponding simply-connected compact Lie group is $SU(n+1)$ and all other Lie groups of type A_n are of the form $U = SU(n+1)/\Gamma$, where $\Gamma \leq Z(SU(n+1)) \simeq \mathbb{Z}_{n+1}$. The integral lattice of U with respect to any bi-invariant metric is given by

$$(A.6) \quad \Lambda_I(SU(n+1)/\Gamma) = \langle kL_1 \rangle + \Lambda_{R^\vee},$$

where $k = 0, 1, \dots, n$ is the smallest generator of Γ .

A.2.2. Type B_n . As a vector space we have $V = \mathbb{R}^n$. The corresponding root system is $R = \{\pm\epsilon_\mu, \pm\epsilon_\mu \pm \epsilon_\nu : 1 \leq \mu \neq \nu \leq n, \pm \text{ independent}\}$ with co-root system $R^\vee = \{\pm 2e_\mu, \pm e_\mu \pm e_\nu : 1 \leq \mu \neq \nu \leq n, \pm \text{ independent}\}$. For a Weyl chamber we choose $\mathcal{C} = \{v \in V : \epsilon_\nu(v) > \epsilon_{\nu+1}(v), 1 \leq \nu \leq n-1, \epsilon_n(v) > 0\}$, which gives the corresponding positive roots $R^+ = \{\epsilon_\mu : 1 \leq \mu \leq n\} \cup \{\epsilon_\mu \pm \epsilon_\nu : 1 \leq \mu < \nu \leq n\}$ and positive basis $B = \{\epsilon_\nu - \epsilon_{\nu+1}, \epsilon_n : 1 \leq \nu \leq n-1\}$. The co-root lattice, central lattice, center and sum of the positive roots (i.e., 2ρ) of this root system are given by

- $\Lambda_{R^\vee} = \langle 2e_\mu, e_\mu \pm e_\nu : 1 \leq \mu \neq \nu \leq n \rangle$
- $\Lambda_Z = \langle e_1, \dots, e_n \rangle = \langle e_1, e_1 - e_2, \dots, e_1 - e_n \rangle$
- $Z = \Lambda_Z / \Lambda_{R^\vee} = \langle \bar{e}_1 \rangle \simeq \mathbb{Z}_2$
- $2\rho \equiv \sum_{\alpha \in R^+} \alpha = \sum_{\nu=1}^n (2n - 2\nu + 1)\epsilon_\nu$

The corresponding simply-connected compact Lie group is $\text{Spin}(2n+1)$ and all Lie groups of type B_n are of the form $U = \text{Spin}(2n+1)/\Gamma$, where $\Gamma \leq Z(\text{Spin}(2n+1)) = \langle \bar{e}_1 \rangle \simeq \mathbb{Z}_2$. The integral lattice of U with respect to any bi-invariant metric is given by

$$(A.7) \quad \Lambda_I(\text{Spin}(2n+1)/\Gamma) = \begin{cases} \Lambda_{R^\vee} & \Gamma \text{ trivial} \\ \Lambda_Z & \Gamma \simeq \mathbb{Z}_2 \end{cases}$$

A.2.3. Type C_n . As a vector space we have $V = \mathbb{R}^n$. The corresponding root system is $R = \{\pm 2\epsilon_\mu, \pm\epsilon_\mu \pm \epsilon_\nu : 1 \leq \mu \neq \nu \leq n, \pm \text{ independent}\}$ with co-root system $R^\vee = \{\pm e_\mu, \pm e_\mu \pm e_\nu : 1 \leq \mu \neq \nu \leq n, \pm \text{ independent}\}$. For a Weyl chamber we choose $\mathcal{C} = \{v \in V : \epsilon_1(v) > \epsilon_2(v) > \dots > \epsilon_n(v) > 0\}$, which gives the corresponding positive roots $R^+ = \{2\epsilon_\mu : 1 \leq \mu \leq n\} \cup \{\epsilon_\mu \pm \epsilon_\nu : 1 \leq \mu < \nu \leq n\}$ and positive basis $B = \{\epsilon_1 - \epsilon_2, \dots, \epsilon_{n-1} - \epsilon_n, 2\epsilon_n\}$. The co-root lattice, central lattice, center and sum of the positive roots (i.e., 2ρ) of this root system are given by

- $\Lambda_{R^\vee} = \langle e_1, \dots, e_n \rangle$
- $\Lambda_Z = \langle e_1, \dots, e_n, F \equiv \frac{1}{2} \sum_{\mu=1}^n e_\mu \rangle = \{(\frac{c_1}{2}, \dots, \frac{c_n}{2}) : c_j \in \mathbb{Z} \text{ and } c_j \equiv c_i \pmod{2}\}$
- $Z = \Lambda_Z / \Lambda_{R^\vee} = \langle \bar{F} \rangle \simeq \mathbb{Z}_2$
- $2\rho \equiv \sum_{\alpha \in R^+} \alpha = \sum_{\nu=1}^n 2(n - \nu + 1)\epsilon_\nu$

The corresponding simply-connected Lie group is $\text{Sp}(n)$ and all other Lie groups of type C_n are of the form $U = \text{Sp}(n)/\Gamma$, where $\Gamma \leq Z(\text{Sp}(n)) = \langle \bar{F} \rangle \simeq \mathbb{Z}_2$. The integral lattice of U with respect to any bi-invariant metric is given by

$$(A.8) \quad \Lambda_I(\mathrm{Sp}(n)/\Gamma) = \begin{cases} \Lambda_{R^\vee} & , \text{ when } \Gamma \simeq 1 \\ \Lambda_Z & , \Gamma = \langle \overline{F} \rangle \simeq \mathbb{Z}_2 \end{cases}$$

A.2.4. *Type D_n .* As a vector space we have $V = \mathbb{R}^n$. The corresponding root system is $R = \{\pm e_\mu \pm e_\nu : 1 \leq \mu \neq \nu \leq n, \pm \text{ independent}\}$ with co-root system $R^\vee = \{\pm e_\mu \pm e_\nu : 1 \leq \mu \neq \nu \leq n, \pm \text{ independent}\}$. For a Weyl chamber we choose $\mathcal{C} = \{v \in V : \epsilon_1(v) > \epsilon_2(v) > \cdots > \epsilon_{n-1}(v) > |\epsilon_n(v)|\}$, which gives the corresponding positive roots $R^+ = \{e_\mu \pm e_\nu : 1 \leq \mu < \nu \leq n\}$ and positive basis $B = \{\epsilon_1 - \epsilon_2, \dots, \epsilon_{n-1} - \epsilon_n, \epsilon_{n-1} + \epsilon_n\}$. The co-root lattice, central lattice, center and sum of the positive roots (i.e., 2ρ) of this root system are given by

- $\Lambda_{R^\vee} = \langle e_\mu \pm e_\nu : 1 \leq \mu < \nu \leq n \rangle$
- $\Lambda_Z = \langle e_1, e_1 - e_2, \dots, e_1 - e_n, F \equiv \frac{1}{2} \sum_{\mu=1}^n e_\mu \rangle = \langle e_1, \dots, e_n, F \rangle$
-

$$Z = \Lambda_Z / \Lambda_{R^\vee} = \langle \bar{e}_1, \overline{F} \rangle \simeq \begin{cases} \mathbb{Z}_4 & n \text{ odd} \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2 & n \text{ even} \end{cases}$$

- $2\rho \equiv \sum_{\alpha \in R^+} \alpha = \sum_{\nu=1}^n 2(n-\nu)\epsilon_\nu$

The corresponding simply-connected Lie group is $\mathrm{Spin}(2n)$ and all other groups of type D_n are of the form $U = \mathrm{Spin}(2n)/\Gamma$, where

$$\Gamma \leq Z(\mathrm{Spin}(2n)) \simeq \begin{cases} \mathbb{Z}_4 & n \text{ odd} \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2 & n \text{ even} \end{cases}$$

In the event that $2n \equiv 2 \pmod{4}$, the integral lattice of U with respect to a bi-invariant metric is

$$(A.9) \quad \Lambda_I(\mathrm{Spin}(2n)/\Gamma) = \begin{cases} \Lambda_{R^\vee} & \Gamma \simeq 1 \\ \langle 2F \rangle + \Lambda_{R^\vee} & \Gamma = \langle 2\overline{F} \rangle \simeq \mathbb{Z}_2 \\ \langle F \rangle + \Lambda_{R^\vee} & \Gamma = \langle \overline{F} \rangle \simeq \mathbb{Z}_4 \end{cases}$$

For $2n \equiv 0 \pmod{4}$, the integral lattice of U with respect to a bi-invariant metric is

$$(A.10) \quad \Lambda_I(\mathrm{Spin}(2n)/\Gamma) = \begin{cases} \Lambda_{R^\vee} & \Gamma \simeq 1 \\ \langle e_1 \rangle + \Lambda_{R^\vee} & \Gamma = \langle \bar{e}_1 \rangle \simeq \mathbb{Z}_2 \oplus 1 \\ \langle F \rangle + \Lambda_{R^\vee} & \Gamma = \langle \overline{F} \rangle \simeq 1 \oplus \mathbb{Z}_2 \\ \langle e_1 + F \rangle + \Lambda_{R^\vee} & \Gamma = \langle \overline{e_1 + F} \rangle \simeq \mathbb{Z}_2 \\ \Lambda_Z & \Gamma = \langle \bar{e}_1, \overline{F} \rangle \simeq \mathbb{Z}_2 \oplus \mathbb{Z}_2 \end{cases}$$

A.2.5. *Type BC_n .* As a vector space we have $V = \mathbb{R}^n$. The corresponding root system is a non-reduced root system which is the union of the root systems of type B_n and C_n : $R = \{\pm\epsilon_\mu, \pm 2\epsilon_\mu, \pm\epsilon_\mu \pm \epsilon_\nu : 1 \leq \mu \neq \nu \leq n\}$ with co-root system $R^\vee = \{\pm 2\epsilon, \pm e_\mu, \pm e_\mu \pm e_\nu : 1 \leq \mu \neq \nu \leq n\}$. For a Weyl chamber we choose $\mathcal{C} = \{v \in V : \epsilon_1(v) > \epsilon_2(v) > \cdots \epsilon_n(v) > 0\}$, which gives the corresponding positive roots $R^+ = \{\epsilon_j, 2\epsilon_j : 1 \leq j \leq n\} \cup \{\epsilon_i \pm \epsilon_j : 1 \leq i < j \leq n\}$ and positive basis $B = \{\epsilon_1 - \epsilon_2, \dots, \epsilon_{n-1} - \epsilon_n, 2\epsilon_n\}$. The co-root lattice, central lattice, center and sum of the positive roots are given by

- $\Lambda_{R^\vee} = \Lambda_{R^\vee}^{C_n} = \langle e_1, \dots, e_n \rangle$
- $\Lambda_Z = \Lambda_Z^{B_n} = \langle e_1, \dots, e_n \rangle$
- $Z = \Lambda_Z / \Lambda_{R^\vee} = 1$
- $2\rho \equiv \sum_{\alpha \in R^+} \alpha = \sum_{j=1}^n (2(n-j) + 3)\epsilon_j$

A.2.6. *Type F_4 .* As a vector space we have $V = \mathbb{R}^4$. The corresponding root system contains the roots coming from B_4 : $R = \{\pm\epsilon_\mu, \pm\epsilon_\mu \pm \epsilon_\nu : 1 \leq \mu \neq \nu \leq 4, \pm \text{independent}\} \cup \{\frac{1}{2} \sum_{\mu=1}^4 \pm\epsilon_\mu : \pm \text{independent}\}$ with co-root system $R^\vee = \{\pm 2e_\mu, \pm e_\mu \pm e_\nu : 1 \leq \mu \neq \nu \leq 4, \pm \text{independent}\} \cup \{\frac{1}{2} \sum_{\mu=1}^4 \pm e_\mu : \pm \text{independent}\}$. For a Weyl chamber \mathcal{C} we choose the component of $V - \cup_{\alpha \in R} \ker(\alpha)$ containing the regular vector $(8, 3, 2, 1)$. Then the positive roots are $R^+ = \{\epsilon_\mu : 1 \leq \mu \leq n\} \cup \{\epsilon_\mu \pm \epsilon_\nu : 1 \leq \mu \leq \nu \leq 4\} \cup \{\frac{1}{2}(\epsilon_1 \pm \epsilon_2 \pm \epsilon_3 \pm \epsilon_4) : \pm \text{independent}\}$ and positive basis $B = \{\alpha_1 = \frac{1}{2}(\epsilon_1 - \epsilon_2 - \epsilon_3 - \epsilon_4), \alpha_2 = \epsilon_4, \alpha_3 = \epsilon_3 - \epsilon_4, \alpha_4 = \epsilon_2 - \epsilon_3\}$. The co-root lattice, central lattice, center and sum of the positive roots are given by

- $\Lambda_{R^\vee} = \langle \frac{1}{2} \sum_{\mu=1}^4 \pm e_\mu, \pm 2e_\mu, \pm e_\mu \pm e_\nu : 1 \leq \mu \neq \nu \leq 4 \rangle$
- $\Lambda_Z = \Lambda_{R^\vee}$
- $Z = \Lambda_Z / \Lambda_{R^\vee} \simeq 1$
- $2\rho \equiv \sum_{\alpha \in R^+} \alpha = 15\epsilon_1 + 5\epsilon_2 + 3\epsilon_3 + \epsilon_4$

The corresponding simply-connected compact Lie group is also denoted by F_4 and it is the unique group of this type. The integral lattice of this group with respect to any bi-invariant metric given by

$$(A.11) \quad \Lambda_I(F_4) = \Lambda_{R^\vee}.$$

A.2.7. *Type G_2 .* As a vector space we have $V = \{v \in \mathbb{R}^3 : \epsilon_1(v) + \epsilon_2(v) + \epsilon_3(v) = 0\}$. The corresponding root system is given by $R = \{(\epsilon_\mu - \epsilon_\nu) \upharpoonright V : 1 \leq \mu \neq \nu \leq 3\} \cup \{\pm\epsilon_\mu \upharpoonright V : 1 \leq \mu \leq 3\}$, which contains the roots of A_2 , and the associated co-root system is given by $R^\vee = \{e_\mu \pm e_\nu : 1 \leq \mu < \nu \leq 3\} \cup \{\pm(2e_1 - e_2 - e_3), \pm(-e_1 + 2e_2 - e_3), \pm(-e_1 - e_2 + 2e_3)\}$. For a Weyl chamber we choose the component \mathcal{C} of $V - \cup \ker(\alpha)$ containing the regular vector $(3, 2, -5)$, which gives the positive roots $R^+ = \{(\epsilon_1 - \epsilon_2) \upharpoonright V, (\epsilon_1 - \epsilon_3) \upharpoonright V, (\epsilon_2 - \epsilon_3) \upharpoonright V, \epsilon_1 \upharpoonright V, \epsilon_2 \upharpoonright V, -\epsilon_3 \upharpoonright V\}$ and, since $\epsilon_1 \upharpoonright V = \frac{1}{3}(2\epsilon_1 - \epsilon_2 - \epsilon_3) \upharpoonright V$, a positive basis $B = \{(\epsilon_1 - \epsilon_2) \upharpoonright V, \epsilon_2 \upharpoonright V\}$. The co-root lattice, central lattice, center and sum of the positive roots are given by

- $\Lambda_{R^\vee} = \langle 2e_1 - e_2 - e_3, -e_1 + 2e_2 - e_3, -e_1 - e_2 + 2e_3, e_\mu \pm e_\nu : 1 \leq \mu < \nu \leq 3 \rangle$
- $\Lambda_Z = \Lambda_{R^\vee}$

- $Z = \Lambda_Z / \Lambda_{R^\vee} \simeq 1$
- $2\rho \equiv \sum_{\alpha \in R^+} \alpha = (3\epsilon_1 + \epsilon_2 - 3\epsilon_3) \upharpoonright V = (2\epsilon_2 - 4\epsilon_3) \upharpoonright V$

The corresponding simply-connected compact Lie group is also denoted by G_2 and it is the unique group of this type. The integral lattice of this group with respect to any bi-invariant metric given by

$$(A.12) \quad \Lambda_I(G_2) = \Lambda_{R^\vee}.$$

A.2.8. *Type E_8 .* As a vector space we have $V = \mathbb{R}^8$. The corresponding root system is the union of the 112 roots of D_8 with 128 additional roots $R = \{\pm\epsilon_\mu \pm \epsilon_\nu : 1 \leq \mu < \nu \leq 8\} \cup \{\frac{1}{2} \sum_{\mu=1}^8 \pm \epsilon_\mu : \text{there are an even number of minus signs}\} = \{\pm\epsilon_\mu \pm \epsilon_\nu : 1 \leq \mu < \nu \leq 8\} \cup \{\frac{1}{2} \sum_{\mu=1}^8 (-1)^{k_\mu} \epsilon_\mu : k_\mu = 0, 1 \text{ and } \sum k_\mu \equiv 0 \pmod{2}\}$ with co-root system $R^\vee = \{e_\mu \pm e_\nu : 1 \leq \mu \neq \nu \leq 8\} \cup \{\frac{1}{2} \sum_{\mu=1}^8 \pm e_\mu : \text{there are an even number of minus signs}\}$. For a Weyl chamber we choose the component \mathcal{C} of $V - \cup \ker(\alpha)$ containing the vector $(0, 1, 2, 3, 4, 5, 6, 23)$. Then the positive roots are $R^+ = R_1^+ \cup R_2^+ \cup R_3^+$, where

- $R_1^+ = \{\frac{1}{2}(\epsilon_8 + \epsilon_7 + \sum_{\mu=1}^6 (-1)^{k_\mu} \epsilon_\mu : \sum k_\mu \equiv 0 \pmod{2}\}$
- $R_2^+ = \{\frac{1}{2}(\epsilon_8 - \epsilon_7 + \sum_{\mu=1}^6 (-1)^{k_\mu} \epsilon_\mu : \sum k_\mu \equiv 1 \pmod{2}\}$
- $R_3^+ = \{\epsilon_\mu \pm \epsilon_\nu : 1 \leq \mu < \nu \leq 8\}$.

The sets R_1 and R_2 each contain 32 elements, while R_3 contains 56. As a positive basis for this root system we have the set B consisting of the elements $\alpha_1 = \frac{1}{2}(\epsilon_1 + \epsilon_8 - \sum_{\mu=2}^7 \epsilon_\mu)$, $\alpha_2 = e_1 + e_2$, $\alpha_{j+1} = \epsilon_j - \epsilon_{j-1}$ for $j = 2, \dots, 7$. Now, since the roots of D_8 are contained in the roots of E_8 we realize that the central lattice of E_8 is contained in the central lattice of D_8 . The co-root lattice, central lattice, center and sum of the positive roots are then given by

- $\Lambda_{R^\vee} = \Lambda_Z = \langle v_1, \dots, v_7, v_8 \rangle$, where $v_1 = \frac{1}{2}(e_1 + e_8 - \sum_{j=2}^7 e_j)$, $v_2 = e_1 + e_2$ and $v_{j+1} = v_j - v_{j-1}$ for $j = 2, \dots, 7$.
- $Z = \Lambda_Z / \Lambda_{R^\vee} \simeq 1$
- $2\rho \equiv \sum_{\alpha \in R^+} \alpha = 32\epsilon_8 + \sum_{j=1}^7 2(8-j)\epsilon_j$

The corresponding simply-connected compact Lie group is also denoted by E_8 and it is the unique group of this type. The integral lattice of this group with respect to any bi-invariant metric given by

$$(A.13) \quad \Lambda_I(E_8) = \Lambda_{R^\vee}.$$

A.2.9. *Type E_7 .* We will describe this root system in terms of E_8 . Letting v_1, \dots, v_8 and $\alpha_1, \dots, \alpha_8$ be as in A.2.8, the vector space V will be the 7-dimensional subspace of \mathbb{R}^8 spanned by $v_1 = \frac{1}{2}(e_1 + e_8 - \sum_{j=2}^7 e_j)$, $v_2 = e_1 + e_2$, $v_{j+1} = e_j - e_{j-1}$ for $j = 2, \dots, 6$. In other words, V is spanned by $e_1, \dots, e_6, e_7 - e_8$ and consists of the vectors in \mathbb{R}^8 where the e_7 and e_8 coordinate are opposite. The corresponding root system $R \subset V^*$ consists of the roots in E_8 in the span of $\alpha_1, \dots, \alpha_7$. Specifically, we have $R = \{\pm(\epsilon_\mu \pm \epsilon_\nu) \upharpoonright V : 1 \leq \mu < \nu \leq 6\} \cup \{\pm(\epsilon_7 - \epsilon_8) \upharpoonright V\} \cup \{\pm \frac{1}{2}((\epsilon_7 - \epsilon_8) \upharpoonright V + \sum_{j=1}^6 (-1)^{k_j} \epsilon_j) : \sum k_j \equiv 1 \pmod{2}\}$. The co-root system

is $R^\vee = \{\pm(e_\mu \pm e_\nu) : 1 \leq \mu < \nu \leq 6\} \cup \{\pm(e_7 - e_8)\} \cup \{\pm\frac{1}{2}(e_7 - e_8 + \sum_{j=1}^6 (-1)^{k_j} e_j) : \sum k_j \equiv 1 \pmod{2}\}$. For a Weyl chamber we choose the component \mathcal{C} of $V - \cup_{\alpha \in R} \ker(\alpha)$ containing the regular vector $6e_1 + 5e_2 + 4e_3 + 3e_4 + 2e_5 + 1e_6 + 11(e_7 - e_8)$. Then the positive roots are given by the set $R^+ = R_1^+ \cup R_2^+ \cup R_3^+$, where

- $R_1^+ = \{(\epsilon_7 - \epsilon_8) \upharpoonright V\}$
- $R_2^+ = \{(\epsilon_\mu \pm \epsilon_\nu) \upharpoonright V : 1 \leq \mu < \nu \leq 6\}$
- $R_3^+ = \{\frac{1}{2}(\epsilon_7 - \epsilon_8 + \sum_{\mu=1}^6 (-1)^{k_\mu} \epsilon_\mu) : \sum k_\mu \equiv 1 \pmod{2}\}$.

The set R_1^+ has one element, the set R_2^+ contains 30 elements and R_3^+ contains 32 elements. As a positive basis for this root system we have $B = \{\alpha_1, \dots, \alpha_7\}$. The co-root lattice, central lattice, center and sum of the positive roots are then given by

- $\Lambda_{R^\vee} = \langle v_1, \dots, v_6, v_7 \rangle$
- $\Lambda_Z = \langle v_1, \dots, v_6, v_7, F \rangle$, where

$$F = \frac{1}{2}(e_7 - e_8) + e_1 + e_2 + e_3 = -v_1 + \frac{1}{2}v_2 - v_3 - v_4 - \frac{3}{2}v_5 - v_6$$

- $Z = \Lambda_Z / \Lambda_{R^\vee} = \langle \overline{F} \rangle \simeq \mathbb{Z}_2$
- $2\rho \equiv \sum_{\alpha \in R^+} \alpha = \sum_{j=1}^6 2(6-j)\epsilon_j + 17(\epsilon_7 - \epsilon_8) \upharpoonright V$

The associated simply-connected compact Lie group will also be denoted by E_7 and all other groups of this type are of the form $U = E_7/\Gamma$, where $\Gamma \leq Z(E_7) = \langle \overline{F} \rangle \simeq \mathbb{Z}_2$. The integral lattice of U with respect to any bi-invariant metric is

$$(A.14) \quad \Lambda_I(E_7/\Gamma) = \begin{cases} \Lambda_{R^\vee} & \Gamma \simeq 1 \\ \Lambda_Z & \Gamma = \langle \overline{F} \rangle \simeq \mathbb{Z}_2 \end{cases}$$

A.2.10. Type E_6 . As a vector space we take V_6 to be the 6-dimensional subspace of \mathbb{R}^8 spanned by v_1, \dots, v_6 where the v_i 's are as in [A.2.8](#). One can check that in this case V_6 consists of the vectors in V_7 which are orthogonal to $e_6 + e_8$ and we see that V_6 has basis given by $e_1, \dots, e_5, e_6 + e_7 - e_8$. The dual space V_6^* is the linear span of $\alpha_1, \dots, \alpha_6$. The roots in V_6 are given by $R = R_8 \cap V_6$. More explicitly, we obtain $R = \{\pm(\epsilon_\mu \pm \epsilon_\nu) \upharpoonright V : 1 \leq \mu < \nu \leq 5\} \cup \{\pm\frac{1}{2}(\epsilon_6 + \epsilon_7 - \epsilon_8 + \sum_{j=1}^5 (-1)^{k_j} \epsilon_j) : \sum_{j=1}^5 k_j \equiv 1 \pmod{2}\}$ and the co-roots are given by $R^\vee = \{\pm(e_\mu \pm e_\nu) \upharpoonright V : 1 \leq \mu < \nu \leq 5\} \cup \{\pm\frac{1}{2}(e_6 + e_7 - e_8 + \sum_{j=1}^5 (-1)^{k_j} e_j) : \sum_{j=1}^5 k_j \equiv 1 \pmod{2}\}$. We choose our Weyl chamber \mathcal{C} to be the component containing the regular vector $5e_1 + 4e_2 + 3e_3 + 2e_4 + 1e_5 + 6(e_6 + e_7 - e_8)$ with corresponding positive roots $R^+ = R_1^+ \cup R_2^+$, where

- $R_1^+ = \{\epsilon_i \pm \epsilon_j : 1 \leq i < j \leq 5\}$
- $R_2^+ = \{\frac{1}{2}(\epsilon_6 + \epsilon_7 - \epsilon_8 + \sum_{j=1}^5 (-1)^{k_j} e_j) : \sum k_j \equiv 1 \pmod{2}\}$.

One can check that the co-root lattice, central lattice, center and the sum of the positive roots are as follows:

- $\Lambda_{R^\vee} = \langle v_1, \dots, v_5, v_6 \rangle$

- $\Lambda_Z = \langle v_1, \dots, v_5, v_6, F \rangle$, where

$$F = \frac{2}{3}(e_6 + e_7 - e_8) = \frac{2}{3}(-2v_1 - \frac{3}{2}v_2 - \frac{5}{2}v_3 - 3v_4 - 2v_5 - v_6 + v_7)$$

- $Z = \Lambda_Z / \Lambda_{R^+} = \langle \overline{F} \rangle \simeq \mathbb{Z}_3$
- $2\rho \equiv \sum_{\alpha \in R^+} \alpha = \sum_{j=1}^5 2(5-j)\epsilon_j + 8(\epsilon_6 + \epsilon_7 - \epsilon_8)$.

The associated simply-connected compact Lie group will also be denoted by E_6 and all other groups of this type are of the form $U = E_6 / \Gamma$, where $\Gamma \leq Z(E_6) = \langle \overline{F} \rangle \simeq \mathbb{Z}_3$. The integral lattice of U with respect to any bi-invariant metric is

$$(A.15) \quad \Lambda_I(E_6 / \Gamma) = \begin{cases} \Lambda_{R^+} & \Gamma \simeq 1 \\ \Lambda_Z & \Gamma = \langle \overline{F} \rangle \simeq \mathbb{Z}_3. \end{cases}$$

APPENDIX B. CONSTRUCTING AN $\text{Ad}(\Delta K)$ -INVARIANT COMPLEMENT

Let U be a compact simple Lie group with bi-invariant metric g_0 . Now, suppose g is a left-invariant metric on U than is naturally reductive with respect to $G \equiv U \times K$ for some $K \leq U$. We now review the procedure for constructing an $\text{Ad}(\Delta K)$ -invariant complement \mathfrak{p} of $\Delta \mathfrak{K}$ in $\mathfrak{g} = \mathfrak{u} \oplus \mathfrak{K}$ as discussed in [DZ].

To begin we let $A : \mathfrak{K}_0 \rightarrow \mathfrak{K}_0$ denote the g_0 -symmetric endomorphism satisfying $h(X, Y) = g_0(AX, Y)$ for each $X, Y \in \mathfrak{K}_0$. Then as is described in [DZ, p. 9-11] there are two cases to consider:

- (1) α is not an eigenvalue of A and $\alpha_j \neq \alpha$ for each $j = 1, \dots, r$.

In this case we consider the symmetric bi-linear form Q on $\mathfrak{g} \times \mathfrak{K}$ given by

$$Q = \beta g \upharpoonright \mathfrak{u} \oplus 0 + \bar{h} \upharpoonright 0 \oplus \mathfrak{K}_0 + \beta_1 g \upharpoonright 0 \oplus \mathfrak{K}_1 + \dots + \beta_r g \upharpoonright 0 \oplus \mathfrak{K}_r,$$

where $\beta = \alpha$, $\beta_j = \frac{\beta \alpha_j}{\alpha - \alpha_j}$, and $\bar{h}(X, Y) = g_0(\overline{A}X, Y)$ is defined by the g_0 -symmetric endomorphism $\overline{A} : \mathfrak{K}_0 \rightarrow \mathfrak{K}_0$ satisfying $A = \beta \overline{A}(\overline{A} + \beta I)^{-1}$. Q can be seen to be non-degenerate on $\mathfrak{g} \times \mathfrak{K}$ and $\Delta \mathfrak{K}$. We then take \mathfrak{p} to be the Q -orthogonal complement of $\Delta \mathfrak{K}$ which is given by

$$(B.1) \quad \mathfrak{p} = \mathfrak{p}_1 \oplus \mathfrak{q}_0 \oplus \mathfrak{q}_1 \oplus \dots \oplus \mathfrak{q}_r,$$

where

- (a) $\mathfrak{p}_1 = \{(X, 0) : X \in \mathfrak{u}\}$;
- (b) $\mathfrak{q}_0 = \{(\overline{A}X, -\beta X) : X \in \mathfrak{K}_0\}$;
- (c) $\mathfrak{q}_j = \{(\beta_j X, -\beta X) : X \in \mathfrak{K}_j\}$ for $j = 1, \dots, r$.

From this one may conclude that the metric $g_{\alpha, \alpha_1, \dots, \alpha_r, h}$ is naturally reductive.

(2) α is an eigenvalue of A or $\alpha_j = \alpha$ for some $j = 1, \dots, r$.

We find the $\text{Ad}(\Delta K)$ -invariant complement \mathfrak{p} of $\Delta\mathfrak{K}$ in $\mathfrak{u} \times \mathfrak{K}$ by considering a proper subgroup $K' \leq K$ with respect to which the metric $g_{\alpha, \alpha_1, \dots, \alpha_r, h}$ falls into the previous case. Indeed, consider the Lie algebra

$$\mathfrak{K}'' = \mathfrak{K}_0'' \oplus (\oplus_{\alpha_j = \alpha} \mathfrak{K}_j),$$

where $\mathfrak{K}_0'' = \{X \in \mathfrak{K}_0 : AX = \alpha X\}$. Then we let \mathfrak{K}' denote the g_0 -orthogonal complement of \mathfrak{K}'' in \mathfrak{K} and let K' denote the corresponding connected proper subgroup of K . One can check that

$$\mathfrak{K}' = \mathfrak{K}_0' \oplus (\oplus_{\alpha_j \neq \alpha} \mathfrak{K}_j),$$

where \mathfrak{K}_0' is the g_0 -orthogonal complement of \mathfrak{K}_0'' in \mathfrak{K}_0 . We can then view the metric $g_{\alpha, \alpha_1, \dots, \alpha_r, h}$ as being induced by the inner product

$$\alpha g_0 \upharpoonright \mathfrak{m}' \oplus h \upharpoonright \mathfrak{K}_0' \oplus (\oplus_{\alpha_j \neq \alpha} \alpha_j g_0 \upharpoonright \mathfrak{K}_j),$$

where $\mathfrak{m}' = \mathfrak{m} \oplus \mathfrak{K}''$ is the g_0 orthogonal complement of \mathfrak{K}' . The metric then falls into the previous case with respect to K' and we take \mathfrak{p} to be the corresponding complement of $\Delta\mathfrak{K}'$ in $\mathfrak{g} \times \mathfrak{K}'$:

$$(B.2) \quad \mathfrak{p} = \mathfrak{p}_1' \oplus \mathfrak{q}_0' \oplus (\oplus_{\alpha_j \neq \alpha} \mathfrak{q}_j),$$

where

- (a) $\mathfrak{p}_1' = \{(X, 0) : X \in \mathfrak{m}'\}$;
- (b) $\mathfrak{q}_0' = \{(\overline{A}X, -\beta X) : X \in \mathfrak{K}_0'\}$;
- (c) $\mathfrak{q}_j = \{(\beta_j X, -\beta X) : X \in \mathfrak{K}_j\}$ for $j = 1, \dots, r$.

However, one can check that \mathfrak{p} is also an $\text{Ad}(\Delta K)$ -invariant complement of $\Delta\mathfrak{K}$ in $\mathfrak{u} \times \mathfrak{K}$ and we can then see that the metric is naturally reductive with respect to $U \times K$.

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